Linear Matrix Inequalities in Control

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Preface

In recent years, linear matrix inequalities (LMI's) have emerged as a powerful tool to approach control problems that appear hard if not impossible to solve in an analytic fashion. Although the history of linear matrix inequalities goes back to the fourties with a major emphasis of their role in control in the sixties through the work of Kalman, Yakubovich, Popov and Willems, only during the last decades powerful numerical interior point techniques have been developed to solve LMI's in a practically efficient manner (Nesterov, Nemirovskii 1994). Today, several commercial and non-commercial software packages are available that allow for simple codings of general control problems into well defined classes of optimization problems. These optimization classes include, for example, linear and quadratic programs, semi-definite programs, quadratic second order cone optimizations, sum-of-squares programs and robust optimizations.

Boosted by the availability of fast and efficient LMI solvers, research in robust control theory has experienced a significant paradigm shift. Instead of arriving at an analytical solution of an optimal control problem and implementing such a solution in software so as to synthesize optimal controllers, today a substantial body of research is devoted to reformulating a control problem to the question whether a specific linear matrix inequality is solvable or, alternatively, to optimizing functionals over linear matrix inequality constraints.

This book aims at providing a state of the art treatment of the theory and the usage and applications of linear matrix inequalities in the general area of systems and control. The main emphasis of this book is to reveal the basic principles and background for formulating desired properties of a control system in the form of linear matrix inequalities, and to demonstrate the techniques to reduce the corresponding controller synthesis problem to an LMI problem. The power of this approach is illustrated by several fundamental robustness and performance problems in analysis and design of linear control systems.

This book has been written as lecture material for a graduate course on the subject of LMI's in systems and control. Within the graduate program of the Dutch Institute of Systems and Control (DISC), this course is intended to provide up-to-date information on the topic for students involved in either the practical or theoretical aspects of control system design. DISC courses have the format of two class hours that are taught once per week during a period of eight weeks. Within the DISC graduate program, the first course on LMI's in control has been given by the authors of this book in 1997. The course has been part of the DISC graduate program since. In addition, the material

has been taught on a regular basis as part of the Hybrid Control (HYCON) graduate school in the European Embedded Control Institute (EECI) in Paris.

Various draft versions of this book have been distributed on the internet as lecture notes to the students following these courses and as a service to the international research community on systems and control. The lecture notes have been slowly evaluating to the present book, thanks to the truly many suggestions, fierce criticism, positive feedback, numerous corrections, encouragements and help of many students and researchers who followed the courses or otherwise read the material. We are very thankful for all the suggestions that helped to improve the manuscript.

Readers of this book are supposed to have an academic background in linear algebra, basic calculus, and possibly in system and control theory.

Chapter 1

Convex optimization and linear matrix inequalities

1.1 Introduction

Optimization questions and decision making processes are abundant in daily life and invariably involve the selection of the best decision from a number of options or a set of candidate decisions. Many examples of this theme can be found in technical sciences such as electrical, mechanical and chemical engineering, in architecture and in economics, but also in the social sciences, in biological and ecological processes, politics and organizational questions. For example, production processes in industry are becoming more and more market driven and require an ever increasing flexibility of product changes and product specifications due to customer demands on quality, price and specification. Products need to be manufactured within strict product specifications, with large variations of component qualities, against competitive prices, with minimal waste of resources, energy and valuable production time, with a minimal time-to-market, subject to safety and security regulations and, of course, at the same time with maximal economical profit. Important economical benefits can therefore only be realized by making proper decisions in the operating conditions of production processes. Consequently, there is a constant need for further optimization, for increased efficiency and a better control of processes. A proper combination of control system design and robust optimization are among the key tools to resolve these questions. This is the main theme of the present book.

Casting an optimization problem in mathematics involves the specification of all candidate decisions and, most importantly, the formalization of the concept of *best* or *optimal decision*. If the universum of all possible decisions in an optimization problem is denoted by a set $\mathscr{X}$, then the set of *feasible (or candidate) decisions* is a subset $\mathscr{S}$ of $\mathscr{X}$ from which the best candidate decision needs to be selected. One approach to quantify the quality of a feasible decision $x \in \mathscr{S}$ is to express its value in terms of a single real quantity $f(x)$ where f is some real valued function $f : \mathscr{S} \to \mathbb{R}$ called

the *objective function* or *cost function*. The value of decision $x \in \mathcal{S}$ is then given by $f(x)$ which quantifies the quality or confidence in this particular decision. Depending on the interpretation of the objective function, we may wish to minimize or maximize f over all feasible candidates in $\mathcal{S}$. An optimal decision is then simply an element of $\mathcal{S}$ that minimizes or maximizes f over all feasible alternatives.

The optimization problem to *minimize* the objective function f over a set of feasible decisions $\mathcal{S}$ involves various specific questions:

(a) What is the least possible cost? That is, determine the *optimal value*

$$V_{\text{opt}} := \inf_{x \in \mathcal{S}} f(x) = \inf\{f(x) \mid x \in \mathcal{S}\}.$$

By convention, the optimal value $V_{\text{opt}} = +\infty$ if $\mathcal{S}$ is empty, while the problem is said to be *unbounded* if $V_{\text{opt}} = -\infty$.

(b) How to determine an *almost optimal solution*, i.e., for arbitrary $\varepsilon > 0$, how to determine $x_\varepsilon \in \mathcal{S}$ such that

$$V_{\text{opt}} \leq f(x_\varepsilon) \leq V_{\text{opt}} + \varepsilon.$$

(c) Does there exist an *optimal solution* $x_{\text{opt}} \in \mathcal{S}$ with $f(x_{\text{opt}}) = V_{\text{opt}}$? If so, we say that the *minimum is attained* and we write $f(x_{\text{opt}}) = \min_{x \in \mathcal{S}} f(x)$.

(d) How to find one, or all, optimal solutions $x_{\text{opt}} \in \mathcal{S}$, if they exist. The set of all optimal solutions is denoted by $\operatorname{argmin}_{x \in \mathcal{S}} f(x)$.

We will address each of these questions as a recurrent theme in this book.

1.2 Facts from convex analysis

In view of the optimization problems just formulated, we are interested in finding conditions for optimal solutions to exist. It is therefore natural to resort to a branch of analysis which provides such conditions: convex analysis. The results and definitions in this subsection are mainly basic, but they have very important implications and applications as we will see later.

We start with summarizing some definitions and elementary properties from linear algebra and functional analysis. We assume the reader to be familiar with the basic concepts of vector spaces, norms and normed linear spaces.

1.2.1 Continuity and compactness

Suppose that $\mathscr{X}$ and $\mathscr{Y}$ are two normed linear spaces. A function f which maps $\mathscr{X}$ to $\mathscr{Y}$ is said to be *continuous at $x_0 \in \mathscr{X}$* if, for every $\varepsilon > 0$, there exists a $\delta = \delta(x_0, \varepsilon)$ such that

$$\|f(x) - f(x_0)\| < \varepsilon \quad \text{whenever} \quad \|x - x_0\| < \delta. \tag{1.2.1}$$

The function f is called *continuous* if it is continuous at all $x_0 \in \mathscr{X}$. Finally, f is said to be *uniformly continuous* if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$, not depending on x_0, such that (1.2.1) holds. Obviously, continuity depends on the definition of the norm in the normed spaces $\mathscr{X}$ and $\mathscr{Y}$. We remark that a function $f : \mathscr{X} \to \mathscr{Y}$ is continuous at $x_0 \in \mathscr{X}$ if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \in \mathscr{X}$, which converges to x_0 as $n \to \infty$, there holds that $f(x_n) \to f(x_0)$.

Now let $\mathscr{S}$ be a subset of the normed linear space $\mathscr{X}$. Then $\mathscr{S}$ is called *compact* if for every sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathscr{S}$ there exists a subsequence $\{x_{n_m}\}_{m=1}^{\infty}$ which converges to an element $x_0 \in \mathscr{S}$. Compact sets in finite dimensional vector spaces are easily characterized. Indeed, if $\mathscr{X}$ is finite dimensional then a subset $\mathscr{S}$ of $\mathscr{X}$ is compact if and only if $\mathscr{S}$ is closed and bounded[1].

The well-known Weierstrass theorem provides a useful tool to determine whether an optimization problem admits a solution. It provides an answer to the third question raised in the previous subsection for special sets $\mathscr{S}$ and special performance functions f.

Proposition 1.1 (Weierstrass) *If $f : \mathscr{S} \to \mathbb{R}$ is a continuous function defined on a compact subset $\mathscr{S}$ of a normed linear space $\mathscr{X}$, then there exists $x_{\min}, x_{\max} \in \mathscr{S}$ such that*

$$f(x_{\min}) = \inf_{x \in \mathscr{S}} f(x) \leq f(x) \leq \sup_{x \in \mathscr{S}} f(x) = f(x_{\max})$$

for all $x \in \mathscr{S}$.

Proof. Define $V_{\min} := \inf_{x \in \mathscr{S}} f(x)$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathscr{S}$ such that $f(x_n) \to V_{\min}$ as $n \to \infty$. As $\mathscr{S}$ is compact, there must exist a subsequence $\{x_{n_m}\}_{m=1}^{\infty}$ of $\{x_n\}$ which converges to an element, say $x_{\min}$, which lies in $\mathscr{S}$. Obviously, $f(x_{n_m}) \to V_{\min}$ and the continuity of f implies that $f(x_{n_m}) \to f(x_{\min})$ as $n_m \to \infty$. We claim that $V_{\min} = f(x_{\min})$. By definition of $V_{\min}$, we have $V_{\min} \leq f(x_{\min})$. Now suppose that the latter inequality is strict, i.e., suppose that $V_{\min} < f(x_{\min})$. Then $0 < f(x_{\min}) - V_{\min} = \lim_{n_m \to \infty} f(x_{n_m}) - \lim_{n_m \to \infty} f(x_{n_m}) = 0$, which yields a contradiction. The proof of the existence of a maximizing element is similar. ∎

> Following his father's wishes, Karl Theodor Wilhelm Weierstrass (1815-1897) studied law, finance and economics at the university of Bonn. His primary interest, however, was in mathematics which led to a serious conflict with his father. He started his career as a teacher of mathematics. After various positions and invitations, he accepted a chair at the 'Industry Institute' in Berlin in 1855. Weierstrass contributed to the foundations of analytic functions, elliptic functions, Abelian functions, converging infinite products, and the calculus of variations. Hurwitz and Frobenius were among his students.

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[1] A set $\mathscr{S}$ is *bounded* if there exists a number B such that for all $x \in \mathscr{S}$, $\|x\| \leq B$; it is *closed* if $x_n \to x$ implies that $x \in \mathscr{S}$.

1.2.2   Convex sets

Proposition 1.1 does not give a constructive method to find the extremal solutions $x_{\min}$ and $x_{\max}$. It only guarantees the existence of these elements for continuous functions defined on compact sets. For many optimization problems these conditions (continuity and compactness) turn out to be overly restrictive. We will therefore resort to more general feasibility sets.

Definition 1.2 (Convex sets) A set $\mathscr{S}$ in a linear vector space is said to be *convex* if

$$\{x_1, x_2 \in \mathscr{S}\} \implies \{x := \alpha x_1 + (1 - \alpha)x_2 \in \mathscr{S} \text{ for all } \alpha \in (0,1)\}.$$

In geometric terms, this states that a convex set is characterized by the property that the line segment connecting any two points of the set, belongs to the set. In general, the *empty set* and *singletons* (sets that consist of one point only) are considered to be convex. The point $\alpha x_1 + (1 - \alpha)x_2$ with $\alpha \in (0,1)$ is called a *convex combination* of the two points x_1 and x_2. More generally, convex combinations are defined for any finite set of points as follows.

Definition 1.3 (Convex combinations) Let $\mathscr{S}$ be a subset of a vector space. The point

$$x := \sum_{i=1}^{n} \alpha_i x_i$$

is called a *convex combination* of $x_1, \ldots, x_n \in \mathscr{S}$ if $\alpha_i \geq 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \alpha_i = 1$.

It is easy to see that the set of all convex combinations of n points $x_1, \ldots, x_n$ in $\mathscr{S}$ is itself convex, i.e.,

$$\mathscr{C} := \{x \mid x \text{ is a convex combination of } x_1, \ldots, x_n\}$$

is convex.

We next define the notion of interior points and closure points of sets. Let $\mathscr{S}$ be a subset of a normed space $\mathscr{X}$. The point $x \in \mathscr{S}$ is called an *interior point* of $\mathscr{S}$ if there exists an $\varepsilon > 0$ such that all points $y \in \mathscr{X}$ with $\|x - y\| < \varepsilon$ also belong to $\mathscr{S}$. The *interior* of $\mathscr{S}$ is the collection of all interior points of $\mathscr{S}$. $\mathscr{S}$ is said to be *open* if it is equal to its interior. The point $x \in \mathscr{X}$ is called a *closure point* of $\mathscr{S}$ if, for all $\varepsilon > 0$, there exists a point $y \in \mathscr{S}$ with $\|x - y\| < \varepsilon$. The *closure* of $\mathscr{S}$ is the collection of all closure points of $\mathscr{S}$. $\mathscr{S}$ is said to be *closed* if it is equal to its closure.

We summarize some elementary properties pertaining to convex sets in the following proposition.

Proposition 1.4 *Let $\mathscr{S}$ and $\mathscr{T}$ be convex sets in a normed vector space $\mathscr{X}$. Then*

(a) *the set $\alpha \mathscr{S} := \{x \mid x = \alpha s, s \in \mathscr{S}\}$ is convex for any scalar α.*

(b) *the sum $\mathscr{S} + \mathscr{T} := \{x \mid x = s + t, s \in \mathscr{S}, t \in \mathscr{T}\}$ is convex.*

(c) for any $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$, $(\alpha_1 + \alpha_2)\mathscr{S} = \alpha_1\mathscr{S} + \alpha_2\mathscr{S}$.

(d) the closure and the interior of $\mathscr{S}$ (and $\mathscr{T}$) are convex.

(e) for any linear transformation $T : \mathscr{X} \to \mathscr{X}$, the image $T\mathscr{S} := \{x \mid x = Ts, s \in \mathscr{S}\}$ and the inverse image $T^{-1}\mathscr{S} := \{x \mid Tx \in \mathscr{S}\}$ are convex.

(f) the intersection $\mathscr{S} \cap \mathscr{T} := \{x \mid x \in \mathscr{S} \text{ and } x \in \mathscr{T}\}$ is convex.

The distributive property in the third item is non trivial and depends on the convexity of $\mathscr{S}$. The last property actually holds for the intersection of an *arbitrary collection* of convex sets, i.e., if $\mathscr{S}_\alpha$, with $\alpha \in A$, A an arbitrary index set, is a family of convex sets then the intersection $\cap_{\alpha \in A}\mathscr{S}_\alpha$ is also convex. This property turns out to be very useful in constructing the smallest convex set that contains a given set.

To give some examples, let a be a non-zero vector in $\mathbb{R}^n$ and $b \in \mathbb{R}$. The *hyperplane* $\{x \in \mathbb{R}^n \mid a^\top x = b\}$ and the *half-space* $\{x \in \mathbb{R}^n \mid a^\top x \leq b\}$ are convex. A *polyhedron* is, by definition, the intersection of finitely many hyperplanes and half-spaces and is convex by the last item of Proposition 1.4. A *polytope* is a compact polyhedron.

Definition 1.5 (Convex hull) The *convex hull* conv$\mathscr{S}$ of any subset $\mathscr{S} \subset \mathscr{X}$ is the intersection of all convex sets containing $\mathscr{S}$. If $\mathscr{S}$ consists of a finite number of elements, then these elements are referred to as the *vertices* or the *generators* of conv$\mathscr{S}$.

It is easily seen that the convex hull of a finite set of points is a polytope. Interestingly, the converse is also true: any polytope is the convex hull of a finite set. Hence, any polytope can be generated as the convex hull of a finite number of points. Since convexity is a property that is closed under intersection, the following proposition is immediate.

Proposition 1.6 (Convex hulls) *For any subset $\mathscr{S}$ of a linear vector space $\mathscr{X}$, the convex hull* conv$(\mathscr{S})$ *is convex and consists precisely of all convex combinations of the elements of $\mathscr{S}$.*

At a few occasions we will need the concept of a cone. A subset $\mathscr{S}$ of a vector space $\mathscr{X}$ is called a *cone* if $\alpha x \in \mathscr{S}$ for all $x \in \mathscr{S}$ and $\alpha > 0$. A *convex cone* is a cone which is a convex set. Like in Proposition 1.4, if $\mathscr{S}$ and $\mathscr{T}$ are convex cones then so are $\alpha\mathscr{S}$, $\mathscr{S} + \mathscr{T}$, $\mathscr{S} \cap \mathscr{T}$, $T\mathscr{S}$ and $T^{-1}\mathscr{S}$ for all scalars α and all linear transformations T. Likewise, the intersection of an arbitrary collection of convex cones is a convex cone again. Important examples of convex cones are defined in terms of inequalities as follows. If the normed space $\mathscr{X}$ is equipped with an inner product $\langle \cdot, \cdot \rangle$, then for a given collection of points $x_1, \ldots, x_n \in \mathscr{X}$ the set

$$\mathscr{S} := \{x \in \mathscr{X} \mid \langle x, x_i \rangle \leq 0 \text{ for } i = 1, \ldots, n\}$$

is a (closed) convex cone. Thus, solution sets of systems of linear inequalities define convex cones.

1.2.3 Convex functions

In mathematics, inequalities are binary relations defined on a set with the purpose to order or sequence its elements. The symbol $\leq$ defines the familiar binary relation 'smaller than or equal to' on the set $\mathbb{R}$ of real numbers which, in fact, makes $\mathbb{R}$ a totally ordered set. In considering cost functions $f : \mathscr{S} \to \mathbb{R}$ the familiar ordering $\leq$ on $\mathbb{R}$ certainly suffices to introduce and analyze the convexity of f. However, since vector and matrix valued functions play a vital role throughout this book, it is useful to introduce a less common but much more general notion of convexity of functions.

It is for this reason that we start the discussion of convex functions with the introduction of the binary relations $\prec$, $\preccurlyeq$, $\succ$ and $\succcurlyeq$ on the sets of symmetric and Hermitian matrices. A complex valued matrix A is *Hermitian* if it is square and $A = A^* = \bar{A}^\top$ where the bar denotes taking complex conjugate of each entry in A. If A is real then this amounts to saying that $A = A^\top$ in which case A is said to be *symmetric*. The sets of all $n \times n$ Hermitian and symmetric matrices will be denoted by $\mathbb{H}^n$ and $\mathbb{S}^n$, respectively, and we will omit the superscript n if the dimension is not relevant for the context. With $n = 1$, $\mathbb{H}^1$ and $\mathbb{S}^1$ simply coincide with the sets of complex and real numbers which, as usual, are identified with the scalar fields $\mathbb{C}$ and $\mathbb{R}$, respectively. The sets $\mathbb{H}^n$ and $\mathbb{S}^n$ naturally become vector spaces when equipped with the usual notion of addition and scalar multiplication of matrices.

A Hermitian or symmetric matrix A is *negative definite* if $x^*Ax < 0$ for all non-zero complex vectors x. It is *negative semi-definite* if the inequality is non-strict, that is, if $x^*Ax \leq 0$ for all non-zero complex vectors x. Similarly, A is *positive definite* or *positive semi-definite* if $-A$ is negative or negative semi-definite, respectively. The symbols $\prec$, $\preccurlyeq$, $\succ$ and $\succcurlyeq$ now define binary relations on both $\mathbb{H}^n$ and $\mathbb{S}^n$ (and certainly not on arbitrary matrices!) as follows

$$
\begin{aligned}
A \prec B &\qquad \text{if } A - B \text{ is negative definite}\\
A \preccurlyeq B &\qquad \text{if } A - B \text{ is negative semi-definite}\\
A \succ B &\qquad \text{if } A - B \text{ is positive definite}\\
A \succcurlyeq B &\qquad \text{if } A - B \text{ is positive semi-definite.}
\end{aligned}
$$

With these definitions $\preccurlyeq$ and $\succcurlyeq$ are (non-strict) partial orderings on $\mathbb{H}$ and $\mathbb{S}$. That is, these binary relations satisfy the properties of *reflexivity* ($A \preccurlyeq A$), *anti-symmetry* ($A \preccurlyeq B$, $B \preccurlyeq A$ implies $A = B$) and *transitivity* ($A \preccurlyeq B$, $B \preccurlyeq C$ implies $A \preccurlyeq C$), but not a total ordering. This means that it is *not true* that for any $A, B \in \mathbb{H}^n$ *either* $A \preccurlyeq B$ *or* $A \succcurlyeq B$. Indeed, for the set $\mathbb{H}^1 = \mathbb{C}$ for example, the complex numbers i and $-i$ satisfy neither $i \preccurlyeq -i$ nor $-i \preccurlyeq i$.

Hermitian and symmetric matrices have real eigenvalues. Indeed, a scalar λ is an eigenvalue of $A \in \mathbb{H}^n$ if the equation $Ax = \lambda x$ has a non-zero vector solution x. Pre-multiplying this equation by x^* gives $\lambda x^*x = x^*Ax = x^*A^*x = \bar{\lambda}x^*x$ which shows that $\lambda = \bar{\lambda}$, i.e., λ must be real. We leave it as an exercise to show that $A \prec 0$ if and only if all its eigenvalues are negative. Similarly, $A \preccurlyeq 0$ if and only if A has non-positive eigenvalues.

Definition 1.7 (Convex functions) A (matrix valued) function $F : \mathscr{S} \to \mathbb{H}$ is *convex* if $\mathscr{S}$ is a non-empty convex set and if for all $x_1, x_2 \in \mathscr{S}$ and $\alpha \in (0,1)$ there holds that

$$F(\alpha x_1 + (1-\alpha)x_2) \preccurlyeq \alpha F(x_1) + (1-\alpha)F(x_2). \tag{1.2.2}$$

F is called *strictly convex* if the inequality (1.2.2) with $\preccurlyeq$ replaced by $\prec$ holds for all $x_1, x_2 \in \mathscr{S}$, $x_1 \neq x_2$ and all $\alpha \in (0,1)$.

Everything that is said here about functions $F : \mathscr{S} \to \mathbb{H}$ also applies to symmetric valued functions $F : \mathscr{S} \to \mathbb{S}$ and to real scalar valued functions $f : \mathscr{S} \to \mathbb{R}$. For the latter, the binary relation $\preccurlyeq$ in (1.2.2) coincides with the usual $\leq$. It is important to note that the domain of a convex function is *by definition* a convex set. Simple examples of real-valued convex functions are $f(x) = x^2$ on $\mathbb{R}$, $f(x) = \sin x$ on $[\pi, 2\pi]$ and $f(x) = -\log x$ on $x > 0$. A (matrix valued) function $F : \mathscr{S} \to \mathbb{H}$ is *concave* if $-F$ is convex.

Many operations on convex functions naturally preserve convexity. For example, if F_1 and F_2 are convex functions with domain $\mathscr{S}$ then linear combinations $\alpha_1 F_1 + \alpha_2 F_2 : x \mapsto \alpha_1 F_1(x) + \alpha_2 F_2(x)$ and composite functions $G(F_1)$ are convex for any non-negative numbers α_1, α_2 and non-decreasing[2] functions $G : \mathbb{H} \to \mathbb{H}$.

There is an easy way to obtain convex sets from convex functions. Let $\Gamma \in \mathbb{H}$. A *sublevel set* of a function $F : \mathscr{S} \to \mathbb{H}$ is a set of the form

$$\mathscr{S}_\Gamma := \{x \in \mathscr{S} \mid F(x) \preccurlyeq \Gamma\}.$$

It is immediate that $\mathscr{S}_{\Gamma_1} \subseteq \mathscr{S}_{\Gamma_2}$ whenever $\Gamma_1 \preccurlyeq \Gamma_2$; that is: sublevel sets are non-decreasing functions (in a set theoretic sense) of $\Gamma \in \mathbb{H}$ (with the partial order $\preccurlyeq$ on $\mathbb{H}$). The following proposition will prove very useful.

Proposition 1.8 *If $F : \mathscr{S} \to \mathbb{H}$ is convex then the sublevel set $\mathscr{S}_\Gamma$ is convex for all $\Gamma \in \mathbb{H}$.*

Proof. Suppose F is convex, let $\Gamma \in \mathbb{H}$ and consider $\mathscr{S}_\Gamma$. If $\mathscr{S}_\Gamma$ is empty then the statement is trivial. Suppose therefore that $\mathscr{S}_\Gamma \neq \emptyset$ and let $x_1, x_2 \in \mathscr{S}_\Gamma$, $\alpha \in (0,1)$. Then, $F(x_1) \preccurlyeq \Gamma$, $F(x_2) \preccurlyeq \Gamma$ and the convexity of $\mathscr{S}$ implies that $\alpha x_1 + (1-\alpha)x_2 \in \mathscr{S}$. Convexity of F now yields that

$$F(\alpha x_1 + (1-\alpha)x_2) \preccurlyeq \alpha F(x_1) + (1-\alpha)F(x_2) \preccurlyeq \alpha\Gamma + (1-\alpha)\Gamma = \Gamma$$

i.e., $\alpha x_1 + (1-\alpha)x_2 \in \mathscr{S}_\Gamma$. ∎

Sublevel sets are commonly used in specifying desired behavior of multi-objective control problems. As an example, suppose that $\mathscr{S}$ denotes a class of (closed-loop) transfer functions and let, for $k = 1, \ldots, K$, $f_k : \mathscr{S} \to \mathbb{R}$ be the kth objective function on $\mathscr{S}$. A multi-objective specification amounts to characterizing one or all transfer functions $x \in \mathscr{S}$ for which the design objectives

$$f_1(x) \leq \gamma_1, \quad f_2(x) \leq \gamma_2, \quad \ldots \quad , f_K(x) \leq \gamma_K$$

hold simultaneously. This multi-objective specification amounts to characterizing the sublevel set

$$\mathscr{S}_\Gamma := \{x \in \mathscr{S} \mid F(x) \preccurlyeq \Gamma\}$$

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[2] A function $G : \mathbb{H} \to \mathbb{H}$ is non-decreasing if $G(Y_1) \preccurlyeq G(Y_2)$ whenever $Y_1 \preccurlyeq Y_2$.

where $F := \mathrm{diag}(f_1, \ldots, f_K)$ is a mapping from $\mathscr{S}$ to $\mathbb{S}^K$ and where $\Gamma = \mathrm{diag}(\gamma_1, \ldots \gamma_K)$ belongs to $\mathbb{S}^K$. The design question will be to decide for which Γ this set is non-empty. Proposition 1.8 together with Proposition 1.4 promises the convexity of $\mathscr{S}_\Gamma$ whenever f_k is convex for all k.

We emphasize that it is *not* true that convexity of the sublevel sets $\mathscr{S}_\Gamma$, $\Gamma \in \mathbb{H}$ implies convexity of F. (See Exercise 2 and Exercise 6 in this Chapter). However, the class of functions for which all sublevel sets are convex is that important that it deserves its own name.

Definition 1.9 (Quasi-convex functions) A function $F : \mathscr{S} \to \mathbb{H}$ is *quasi-convex* if its sublevel sets $\mathscr{S}_\Gamma$ are convex for all $\Gamma \in \mathbb{H}$.

It is easy to verify that F is quasi-convex if and only if for all $\alpha \in (0,1)$ and for all $x_1, x_2 \in \mathscr{S}$ we have

$$F(\alpha x_1 + (1-\alpha)x_2) \preccurlyeq \lambda_{\max}(x_1, x_2)I$$

where $\lambda_{\max}(x_1, x_2) = \max[\lambda_{\max}F(x_1), \lambda_{\max}F(x_2)]$ is the maximum of the largest eigenvalues of $F(x_1)$ and $F(x_2)$. In particular, every convex function is quasi-convex.

We conclude this section with the introduction of affine sets and affine functions. A subset $\mathscr{S}$ of a linear vector space is called an *affine set* if the point $x := \alpha x_1 + (1-\alpha)x_2$ belongs to $\mathscr{S}$ for every $x_1 \in \mathscr{S}$, $x_2 \in \mathscr{S}$ and $\alpha \in \mathbb{R}$. The geometric idea is that for any two points of an affine set, also the entire line through these points belongs to the set. From Definition 1.2 it is evident that every affine set is convex. The empty set and singletons are generally considered to be affine. Each non-empty affine set $\mathscr{S}$ in a finite dimensional vector space $\mathscr{X}$ can be written as

$$\mathscr{S} = \{x \in \mathscr{X} \mid x = x_0 + m, m \in \mathscr{M}\}$$

where $x_0 \in \mathscr{X}$ is a vector and $\mathscr{M}$ is a linear subspace of $\mathscr{X}$. That is, affine sets are *translates* of linear subspaces. For any such representation, the linear subspace $\mathscr{M} \subseteq \mathscr{X}$ is uniquely defined, but the vector x_0 is not.

Definition 1.10 (Affine functions) A (matrix valued) function $F : \mathscr{S} \to \mathscr{T}$ is *affine* if

$$F(\alpha x_1 + (1-\alpha)x_2) = \alpha F(x_1) + (1-\alpha)F(x_2)$$

for all $x_1, x_2 \in \mathscr{S}$ and $\alpha \in \mathbb{R}$.

If $\mathscr{S}$ and $\mathscr{T}$ are finite dimensional then any affine function $F : \mathscr{S} \to \mathscr{T}$ can be represented as $F(x) = F_0 + T(x)$ where $F_0 \in \mathscr{T}$ and $T : \mathscr{S} \to \mathscr{T}$ is a linear map. Indeed, setting $F_0 = F(0)$ and $T(x) = F(x) - F_0$ establishes this representation. In particular, $F : \mathbb{R}^n \to \mathbb{R}^m$ is affine if and only if there exists $x_0 \in \mathbb{R}^n$ such that $F(x) = F(x_0) + T(x - x_0)$ where T is a *matrix* of dimension $m \times n$. Note that all affine functions are convex as well as concave.

1.3 Convex optimization

Well prepared with definitions and elementary properties of convex sets and convex functions, we hope that this section will convince the most skeptical reader why convexity of sets and functions is such a desirable property for optimization.

1.3.1 Local and global minima

Anyone who gained experience with numerical optimization methods got familiar with the pitfalls of local minima and local maxima. One crucial reason for studying convex functions is related to the *absence of local minima*.

Definition 1.11 (Local and global optimality) Let $\mathscr{S}$ be a subset of a normed space $\mathscr{X}$. An element $x_0 \in \mathscr{S}$ is said to be a *local optimal solution* of $F : \mathscr{S} \to \mathbb{H}$ if there exists $\varepsilon > 0$ such that

$$F(x_0) \preccurlyeq F(x) \tag{1.3.1}$$

for all $x \in \mathscr{S}$ with $\|x - x_0\| < \varepsilon$. It is called a *global optimal solution* if (1.3.1) holds for all $x \in \mathscr{S}$.

In words, $x_0 \in \mathscr{S}$ is a local optimal solution if there exists a neighborhood of x_0 such that $F(x_0) \preccurlyeq F(x)$ for all feasible points nearby x_0. Note that we defined the notion of local optima for *matrix valued* functions here! According to this definition, a global optimal solution is also locally optimal. Here is a simple and nice result which provides one of our main interests in convex functions.

Proposition 1.12 *Suppose that $F : \mathscr{S} \to \mathbb{H}$ is convex. Every local optimal solution of F is a global optimal solution. Moreover, if F is strictly convex, then the global optimal solution is unique.*

Proof. Let F be convex and suppose that $x_0 \in \mathscr{S}$ is a local optimal solution of F. Then for all $x \in \mathscr{S}$ and $\alpha \in (0,1)$ sufficiently small,

$$F(x_0) \preccurlyeq F(x_0 + \alpha(x - x_0)) = F((1-\alpha)x_0 + \alpha x) \preccurlyeq (1-\alpha)F(x_0) + \alpha F(x). \tag{1.3.2}$$

This implies that

$$0 \preccurlyeq \alpha(F(x) - F(x_0)) \tag{1.3.3}$$

or $F(x_0) \preccurlyeq F(x)$. Since $x \in \mathscr{S}$ is arbitrary, it follows that x_0 is a global optimal solution of F. If F is strictly convex, then the second inequality in (1.3.2) is strict ($\prec$) so that (1.3.3) becomes strict for all $x \in \mathscr{S}$. Hence, x_0 must be unique. ∎

It is very important to emphasize that Proposition 1.12 does not make any statement about the *existence* of optimal solutions $x_0 \in \mathscr{S}$ that minimize F. It merely says that all locally optimal solutions are globally optimal.

Remark 1.13 Proposition 1.12 does not hold for quasi-convex functions.

1.3.2 Uniform bounds

The second reason to investigate convex functions comes from the fact that uniform upperbounds of convex functions can be verified on subsets of their domain. Here are the details: let $\mathscr{S}_0$ be a set and suppose that $F : \mathscr{S} \to \mathbb{H}$ is a function with domain

$$\mathscr{S} = \text{conv}(\mathscr{S}_0).$$

As we have seen in Proposition 1.6, $\mathscr{S}$ is convex and we have the following property which is both simple and powerful.

Proposition 1.14 *Let $F : \mathscr{S} \to \mathbb{H}$ be a convex function where $\mathscr{S} = \text{conv}(\mathscr{S}_0)$ and let $\Gamma \in \mathbb{H}$. Then $F(x) \preccurlyeq \Gamma$ for all $x \in \mathscr{S}$ if and only if $F(x) \preccurlyeq \Gamma$ for all $x \in \mathscr{S}_0$.*

Proof. The 'only if' part is trivial. To see the 'if' part, Proposition 1.6 implies that every $x \in \mathscr{S}$ can be written as a convex combination $x = \sum_{i=1}^{n} \alpha_i x_i$ where $n > 0$, $\alpha_i \geq 0$, $x_i \in \mathscr{S}_0$, $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \alpha_i = 1$. Using convexity of F and non-negativity of the α_i's, we infer

$$F(x) = F(\sum_{i=1}^{n} \alpha_i x_i) \preccurlyeq \sum_{i=1}^{n} \alpha_i F(x_i) \preccurlyeq \sum_{i=1}^{n} \alpha_i \Gamma = \Gamma,$$

which yields the result.   ∎

Proposition 1.14 states that the uniform bound $F(x) \preccurlyeq \Gamma$ on $\mathscr{S}$ can equivalently be verified on the set $\mathscr{S}_0$. This simple observation is of great practical relevance when $\mathscr{S}_0$ contains only a *finite number of elements*, i.e., when $\mathscr{S}$ is a polytope. Proposition 1.14 then states that it requires a *finite number of tests* to conclude whether or not $F(x) \preccurlyeq \Gamma$ for all $x \in \mathscr{S}$.

1.3.3 Duality and convex programs

In many optimization problems, the set of all possible decisions is a real valued finite dimensional vector space $\mathscr{X} = \mathbb{R}^n$ and the space of feasible decisions typically consists of $x \in \mathscr{X}$ that satisfy a finite number of *inequalities* and *equations* of the form

$$\begin{aligned} g_i(x) &\leq 0, & i &= 1, \ldots, k \\ h_j(x) &= 0, & j &= 1, \ldots, l \end{aligned}$$

where g_i and h_j are real-valued functions on $\mathscr{X}$. Indeed, saturation constraints, safety margins, evolution constraints, and a large number of constitutive and balance equations can be written in this way. The space of feasible decisions $\mathscr{S} \subset \mathscr{X}$ can then be expressed as

$$\mathscr{S} := \{x \in \mathscr{X} \mid G(x) \preccurlyeq 0, \quad H(x) = 0\} \tag{1.3.4}$$

where $G : \mathscr{X} \to \mathbb{S}^k$ and $H : \mathscr{X} \to \mathbb{S}^l$ are the symmetric matrix valued functions $G(x) = \text{diag}(g_1(x), \ldots, g_k(x))$ and $H(x) = \text{diag}(h_1(x), \ldots, h_l(x))$. We remark here that the space (1.3.4) is actually a much more

general formalism to represent feasibility sets through the binary relations $\preccurlyeq$ (or possibly $\prec$) when compared to the binary relations $\leq$ and $<$.

With constraints of the form (1.3.4), we consider the optimization problem to find the optimal value

$$P_{\mathrm{opt}} := \inf\{f(x) \mid x \in \mathscr{S}\}$$

and possibly optimal solutions $x_{\mathrm{opt}} \in \mathscr{S}$ such that $f(x_{\mathrm{opt}}) = P_{\mathrm{opt}}$. Here, $f : \mathscr{X} \to \mathbb{R}$ is a given objective function. In this section, we will refer to this constrained optimization problem as a *primal optimization problem* and to P_{opt} as the *primal optimal value*. To make the problem non-trivial, we will assume that $P_{\mathrm{opt}} > -\infty$ and that $\mathscr{S}$ is non-empty.

Remark 1.15 If $\mathscr{X}$, f and G are convex and H is affine, then $\mathscr{S}$ is convex, in which case this problem is commonly referred to as a *convex program*. This is probably the only tractable instance of this problem and its study certainly belongs to the most sophisticated area of nonlinear optimization theory. The Karush-Kuhn-Tucker Theorem, presented below, is the key to understanding convex programs. The special instance where f, G and H are all affine functions makes the problem to determine P_{opt} a *semi-definite program*. If f is quadratic (i.e., f is of the form

$$f(x) = \begin{pmatrix} 1 \\ x \end{pmatrix}^{\top} \begin{pmatrix} q & s \\ s^{\top} & R \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}$$

for some $q \in \mathbb{R}$, $s^{\top} \in \mathbb{R}^n$ and $R = R^{\top} \in \mathbb{S}^n$) and G and H are affine, this is a *quadratic programming problem*.

Obviously, for any $x_0 \in \mathscr{S}$ we have that $P_{\mathrm{opt}} \leq f(x_0)$, i.e., an *upperbound* of P_{opt} is obtained from any feasible point $x_0 \in \mathscr{S}$. On the other hand, to find a *lower bound* of P_{opt}, let us first equip the set $\mathbb{S}$ of symmetric matrices (of any dimension) with the inner product

$$\langle A, B \rangle := \mathrm{trace}(AB), \quad A, B \in \mathbb{S}. \tag{1.3.5}$$

This makes $\mathbb{S}$ an inner product space with corresponding norm $\|A\| = \sqrt{\langle A, A \rangle}$ that is usually referred to as the *Frobenius norm* of a symmetric matrix $A \in \mathbb{S}$.

Suppose that $x \in \mathscr{X}$ satisfies $G(x) \preccurlyeq 0$ and $H(x) = 0$. Then for arbitrary matrices $Y \in \mathbb{S}^k$ with $Y \succcurlyeq 0$ and $Z \in \mathbb{S}^l$ we have that

$$L(x, Y, Z) := f(x) + \langle Y, G(x) \rangle + \langle Z, H(x) \rangle \leq f(x).$$

Here, $L(\cdot, \cdot, \cdot)$ is called a *Lagrangian*, which is a function on $\mathscr{X} \times \mathbb{S}^k \times \mathbb{S}^l$, and we used the fact that $\langle A, B \rangle \leq 0$ if $A \succcurlyeq 0$ and $B \preccurlyeq 0$ (See exercise 14 in this chapter). It is immediate that for all $Y \in \mathbb{S}^k$, $Y \succcurlyeq 0$ and $Z \in \mathbb{S}^l$ we have that

$$\ell(Y, Z) := \inf_{x \in \mathscr{X}} L(x, Y, Z) \leq \inf_{x \in \mathscr{S}} L(x, Y, Z) \leq \inf_{x \in \mathscr{S}} f(x) = P_{\mathrm{opt}}.$$

The function $\ell(\cdot, \cdot)$ is the *Lagrange dual cost*. A key property of the Lagrange dual cost is that it is a concave function of its arguments. Indeed, for any pair of symmetric matrices (Y_1, Z_1) and (Y_2, Z_2)

with $Y_1 \succcurlyeq 0$ and $Y_2 \succcurlyeq 0$ the convex combination $\alpha_1(Y_1, Z_1) + \alpha_2(Y_2, Z_2)$ with $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$ satisfies

$$
\begin{aligned}
\ell(\alpha_1 Y_1 + \alpha_2 Y_2, \alpha_1 Z_1 + \alpha_2 Z_2) &= \inf_{x \in \mathscr{X}} \sum_{i=1}^{2} [\alpha_i f(x) + \alpha_i \langle Y_i, G(x) \rangle + \alpha_i \langle Z_i, H(x) \rangle] \\
&\geq \inf_{x \in \mathscr{X}} \alpha_1 [f(x) + \langle Y_1, G(x) \rangle + \langle Z_1, H(x) \rangle] + \inf_{x \in \mathscr{X}} \alpha_2 [F(x) + \langle Y_2, G(x) \rangle + \langle Z_2, H(x) \rangle] \\
&= \alpha_1 \ell(Y_1, Z_1) + \alpha_2 \ell(Y_2, Z_2)
\end{aligned}
$$

which shows that ℓ is concave, no matter whether or not the primal problem is a convex program. A pair of symmetric matrices (Y, Z) with $Y \succcurlyeq 0$ is said to be *feasible for the dual problem* if $\ell(Y, Z) > -\infty$. Suppose that there exists at least one such feasible pair (Y, Z). Since ℓ is independent of x, we may conclude that

$$
D_{\mathrm{opt}} := \sup_{Y \succcurlyeq 0, Z} \ell(Y, Z) = \sup_{Y \succcurlyeq 0, Z} \inf_{x \in \mathscr{X}} L(x, Y, Z) \leq P_{\mathrm{opt}}
$$

provides a *lower bound* of P_{opt}. Since ℓ is concave the *dual optimization problem* to determine D_{opt} is therefore a concave optimization problem. The main reason to consider this problem is that the constraints in the dual problem are much simpler to deal with than the ones in the primal problem.

Of course, the question arises when $D_{\mathrm{opt}} = P_{\mathrm{opt}}$. To answer this question, suppose that $\mathscr{X}$, f and G are convex and H is affine. As noted before, this implies that $\mathscr{S}$ is convex. We will say that $\mathscr{S}$ satisfies the *constraint qualification* if there exists a point x_0 in the interior of $\mathscr{X}$ with $G(x_0) \preccurlyeq 0$, $H(x_0) = 0$ such that $g_j(x_0) < 0$ for all component functions g_j that are *not* affine[3]. In particular, $\mathscr{S}$ satisfies the constraint qualification if G is affine. We have the following central result.

Theorem 1.16 (Karush-Kuhn-Tucker) *Suppose that $\mathscr{X}$, f and G are convex and H is affine. Assume that $P_{opt} > -\infty$ and $\mathscr{S}$ defined in (1.3.4) satisfies the constraint qualification. Then*

$$
D_{opt} = P_{opt}.
$$

and there exist symmetric matrices $Y_{opt} \in \mathbb{S}^k$, $Y_{opt} \succcurlyeq 0$ and $Z_{opt} \in \mathbb{S}^l$, such that $D_{opt} = \ell(Y_{opt}, Z_{opt})$, i.e., the dual optimization problem admits an optimal solution. Moreover, x_{opt} is an optimal solution of the primal optimization problem and (Y_{opt}, Z_{opt}) is an optimal solution of the dual optimization problem, if and only if

(a) $G(x_{opt}) \preccurlyeq 0$, $H(x_{opt}) = 0$,

(b) $Y_{opt} \succcurlyeq 0$ and x_{opt} minimizes $L(x, Y_{opt}, Z_{opt})$ over all $x \in \mathscr{X}$ and

(c) $\langle Y_{opt}, G(x_{opt}) \rangle = 0$.

The result of Theorem 1.16 is very general and provides a strong tool in convex optimization. This, because the dual optimization problem is, in general, simpler and, under the stated assumptions, admits an optimal solution irrespective of whether the primal optimization problem admits an optimal

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[3]Some authors call $\mathscr{S}$ *superconsistent* and the point x_0 a *Slater point* in that case.

solution. The optimal solutions $(Y_{\text{opt}}, Z_{\text{opt}})$ of the dual optimization problem are generally called *Kuhn-Tucker points*. The conditions 1, 2 and 3 in Theorem 1.16 are called the *primal feasibility*, the *dual feasibility* and the *alignment* (or *complementary slackness*) condition.

Theorem 1.16 provides a conceptual solution of the primal optimization problem as follows. First construct the dual optimization problem to maximize $\ell(Y, Z)$. Second, calculate a Kuhn-Tucker point $(Y_{\text{opt}}, Z_{\text{opt}})$ which defines an optimal solution to the dual problem (existence is guaranteed). Third, determine (if any) the set of optimal solutions $\mathscr{P}'_{\text{opt}}$ which minimize $L(x, Y_{\text{opt}}, Z_{\text{opt}})$ over all $x \in \mathscr{X}$. Fourth, let $\mathscr{P}_{\text{opt}}$ be the set of points $x_{\text{opt}} \in \mathscr{P}'_{\text{opt}}$ such that $G(x_{\text{opt}}) \preccurlyeq 0$, $H(x_{\text{opt}}) = 0$ and $\langle Y_{\text{opt}}, G(x_{\text{opt}}) \rangle = 0$. Then $\mathscr{P}_{\text{opt}}$ is the set of optimal solutions of the primal optimization problem. We emphasize again that optimal solutions to the dual problem are guaranteed to exist, optimal solutions of the primal problem may not exist.

Remark 1.17 In order that the triple $(x_{\text{opt}}, Y_{\text{opt}}, Z_{\text{opt}})$ defined in Theorem 1.16 exists, it is necessary and sufficient that $(x_{\text{opt}}, Y_{\text{opt}}, Z_{\text{opt}})$ be a *saddle point* of the Lagrangian L in the sense that

$$L(x_{\text{opt}}, Y, Z) \leq L(x_{\text{opt}}, Y_{\text{opt}}, Z_{\text{opt}}) \leq L(x, Y_{\text{opt}}, Z_{\text{opt}})$$

for all $x \in \mathscr{X}$, $Y \in \mathbb{S}^k$, $Y \succcurlyeq 0$ and $Z \in \mathbb{S}^l$. In that case,

$$P_{\text{opt}} = L(x_{\text{opt}}, Y_{\text{opt}}, Z_{\text{opt}}) = \inf_x \sup_{Y \succcurlyeq 0, Z} L(x, Y, Z) = \sup_{Y \succcurlyeq 0, Z} \inf_x L(x, Y, Z) =$$

$$= D_{\text{opt}}.$$

That is, the optimal value of the primal and dual optimization problem coincide with the *saddle point value* $L(x_{\text{opt}}, Y_{\text{opt}}, Z_{\text{opt}})$. Under the given conditions, Theorem 1.16 therefore states that x_{opt} is an optimal solution of the primal optimization problem if and only if there exist $(Y_{\text{opt}}, Z_{\text{opt}})$ such that $Y_{\text{opt}} \succcurlyeq 0$ and $(x_{\text{opt}}, Y_{\text{opt}}, Z_{\text{opt}})$ is a saddle point of L.

Remark 1.18 A few generalizations of Theorem 1.16 are worth mentioning. The inequality constraints $G(x) \preccurlyeq 0$ in (1.3.4) can be replaced by the more general constraint $G(x) \in \mathscr{K}$, where $\mathscr{K} \subset \mathscr{H}$ is a closed convex cone in a Hilbert space $\mathscr{H}$. In the definition of the Lagrangian L, the matrices Y and Z define *linear functionals* $\langle Y, \cdot \rangle$ and $\langle Z, \cdot \rangle$ on the inner product spaces $\mathbb{S}^k$ and $\mathbb{S}^l$, respectively. For more general Hilbert spaces $(\mathscr{H}, \langle \cdot, \cdot \rangle)$, the constraint $Y \succcurlyeq 0$ needs to be replaced by the requirement that $\langle Y, G(x) \rangle \leq 0$ for all $x \in \mathscr{X}$ with $G(x) \in \mathscr{K}$. This is equivalent to saying that the linear functional $\langle Y, \cdot \rangle$ is non-positive on the closed and convex cone $\mathscr{K}$. Although interesting, we will not further exploit this more general structure in this book.

Joseph-Louis Lagrange (1736-1813) studied at the College of Turin and he became interested in mathematics when he read a copy of Halley's work on the use of algebra in optics. Although he decided to devote himself to mathematics, he did not have the benefit of studying under supervision of a leading mathematician. Before writing his first paper, he sent his results to Euler, who at that time was working in Berlin. Lagrange worked on the calculus of variations and regularly corresponded on this topic with Euler. Among many contributions in mathematics, Lagrange worked on the calculus of differential equations and applications in fluid mechanics where he first introduced the Lagrangian function.

Figure 1.1: Joseph-Louis Lagrange

1.3.4 Subgradients

Our fourth reason of interest in convex functions comes from the geometric idea that through any point on the graph of a convex function we can draw a line such that the entire graph lies above or on the line. For functions $f : \mathscr{S} \to \mathbb{R}$ with $\mathscr{S} \subseteq \mathbb{R}$, this idea is pretty intuitive from a geometric point of view. The general result is stated in the next Proposition and its proof is a surprisingly simple and beautiful application of Theorem 1.16.

Proposition 1.19 *Let $\mathscr{S} \subseteq \mathbb{R}^n$ and let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^n$. If $f : \mathscr{S} \to \mathbb{R}$ is convex then for all x_0 in the interior of $\mathscr{S}$ there exists a vector $g = g(x_0) \in \mathbb{R}^n$, such that*

$$f(x) \geq f(x_0) + \langle g, x - x_0 \rangle \tag{1.3.6}$$

for all $x \in \mathscr{S}$.

Proof. The set $\mathscr{S}' := \{x \in \mathscr{S} \mid x - x_0 = 0\}$ has the form (1.3.4) and we note that the primal optimal value $P_{\text{opt}} := \inf_{x \in \mathscr{S}'} f(x) - f(x_0) = 0$. Define the Lagrangian $L(x, z) := f(x) - f(x_0) + \langle z, x - x_0 \rangle$ and the corresponding dual optimization $D_{\text{opt}} := \sup_{z \in \mathscr{S}} \inf_{x \in \mathbb{R}^n} L(x, z)$. (In contrast to the previous subsection, the inner product $\langle \cdot, \cdot \rangle$ is now defined on vectors rather than on symmetric matrices). Then $D_{\text{opt}} \leq P_{\text{opt}} = 0$ and since $\mathscr{S}'$ trivially satisfies the constraint qualification, we infer from Theorem 1.16 that there exists $z_{\text{opt}} \in \mathbb{R}^n$ such that

$$D_{\text{opt}} = 0 = \inf_{x \in \mathscr{S}} f(x) - f(x_0) + \langle z_{\text{opt}}, x - x_0 \rangle.$$

Consequently, $f(x) - f(x_0) + \langle z_{\text{opt}}, x - x_0 \rangle \geq 0$ for all $x \in \mathscr{S}$ which yields (1.3.6) by setting $g := -z_{\text{opt}}$.   ∎

A vector g satisfying (1.3.6) is called a *subgradient* of f at the point x_0, and the affine function defined by the right-hand side of (1.3.6) is called a *support functional for f at x_0*. Inequality (1.3.6) is generally referred to as the *subgradient inequality*. We emphasize that the subgradient of a convex function f at a point is in general non-unique. Indeed, the real-valued function $f(x) = |x|$ is convex on $\mathbb{R}$ and has any real number $g \in [-1, 1]$ as its subgradient at $x = 0$. The set of all subgradients of f at x_0 is the *subdifferential of f at x_0* and is denoted by $\partial f(x_0)$ (or $\partial_x f(x_0)$ if the independent variable need to be displayed explicitly). From the subgradient inequality (1.3.6) it is immediate that $\partial(\alpha f)(x) = \alpha \partial f(x)$ for all x and $\alpha > 0$. Also, $x_0 \in \mathscr{S}$ is a global optimal solution of f if and only if $0 \in \partial f(x_0)$. We remark that for a convex function f, $\partial f(x)$ is a closed convex set for any x in the interior of its domain. As a more striking property, let f_1 and f_2 be convex functions with domains $\mathscr{S}_1$ and $\mathscr{S}_2$, respectively, then

$$\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x)$$

for all x in the interior of $\mathscr{S}_1 \cap \mathscr{S}_2$. So, taking subdifferentials is a linear operation in the sense of set additions.

Remark 1.20 Proposition 1.19 gives a necessary condition for convexity of a function f. It can be shown that if the gradient

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{pmatrix}^{\top}$$

exists and is continuous at $x \in \mathscr{S}$ then $\partial f(x) = \nabla f(x)$. So, every gradient is a subgradient. Conversely, if f has a unique subgradient at x, then also the gradient of f exists at x and $\partial f(x) = \nabla f(x)$. One calls f *differentiable* at x in that case.

The geometric interpretation of Proposition 1.19 is that the graphs of the affine functions $x \mapsto f(x_0) + \langle g, x - x_0 \rangle$, with $g \in \partial f(x_0)$, range over the collection of all hyperplanes which are tangent to the graph of f at the point $(x_0, f(x_0))$. That is, the graph of f lies on or above these hyperplanes each of which contains the point $(x_0, f(x_0))$. If we consider the right hand side of (1.3.6), then trivially $\langle g, x - x_0 \rangle > 0$ implies that $f(x) > f(x_0)$. Thus all points in the half-space $\mathscr{H}_+ := \{x \in \mathscr{S} \mid \langle g, x - x_0 \rangle > 0\}$ lead to larger values of f than $f(x_0)$. In particular, subgradients always point in the direction of increasing function values and therefore, in searching for the global minimum of f we can disregard half-spaces of the form $\mathscr{H}_+$.

This last observation is at the basis of the *ellipsoid algorithm*: a simple, numerically robust and straightforward iterative algorithm for the computation of optimal values.

Algorithm 1.21 (Ellipsoid algorithm) Aim: determine the optimal value of a convex function $f : \mathscr{S} \to \mathbb{R}$.

Input: A convex function $f : \mathscr{S} \to \mathbb{R}$ with $\mathscr{S} \subset \mathbb{R}^n$. An ellipsoid

$$\mathscr{E}_0 := \{x \in \mathbb{R}^n \mid (x - x_0)^{\top} P_0^{-1} (x - x_0) \leq 1\}.$$

centered at $x_0 \in \mathbb{R}^n$ and oriented by a positive definite matrix $P_0 = P_0^{\top}$ such that it contains an optimal solution of the problem to minimize f. Let $\varepsilon > 0$ be an accuracy level. Let $k = 0$.

Step 1:   Compute a subgradient $g_k \in \partial f(x_k)$ and set

$$L_k := \max_{\ell \leq k} \left(f(x_\ell) - \sqrt{g_\ell^\top P_\ell g_\ell} \right)$$

$$U_k := \min_{\ell \leq k} f(x_\ell)$$

If $g_k = 0$ or $U_k - L_k < \varepsilon$, then set $x^* = x_k$ and stop. Otherwise proceed to Step 2.

Step 2:   Put $\mathscr{H}_k := \mathscr{E}_k \cap \{x \in \mathbb{R}^n \mid \langle g_k, x - x_k \rangle \leq 0\}$.

Step 3:   Set

$$x_{k+1} := x_k - \frac{P_k g_k}{(n+1)\sqrt{g_k^\top P_k g_k}}$$

$$P_{k+1} := \frac{n^2}{n^2 - 1} \left(P_k - \frac{2}{(n+1)g_k^\top P_k g_k} P_k g_k g_k^\top P_k \right)$$

and define the ellipsoid

$$\mathscr{E}_{k+1} := \{x \in \mathbb{R}^n \mid (x - x_{k+1})^\top P_{k+1}^{-1}(x - x_{k+1}) \leq 1\}$$

with center x_{k+1} and orientation P_{k+1}.

Step 4:   Set k to $k+1$ and return to Step 1.

Output: The point x^* with the property that $|f(x^*) - \inf_{x \in \mathscr{S}} f(x)| \leq \varepsilon$.

The algorithm therefore determines the optimal value of f with arbitrary accuracy. We emphasize that the point x^* is generally *not an optimal or almost optimal solution* unless $g_k = 0$ upon termination of the algorithm. Only in that case x^* is an optimal solution. Hence, the algorithm does not necessarily calculate a solution, but only the optimal value $V_{\text{opt}} = \inf_{x \in \mathscr{S}} f(x)$.

The idea behind the algorithm is as follows. The algorithm is initialized by a 'non-automated' choice of x_0 and P_0 such that there exists an optimal solution x_{opt} in the ellipsoid $\mathscr{E}_0$. If $\mathscr{S}$ is bounded then the safest choice would be such that $\mathscr{S} \subseteq \mathscr{E}_0$. The subgradients $g_k \in \partial f(x_k)$ divide $\mathbb{R}^n$ in the two half-spaces

$$\{x \mid \langle g_k, x - x_k \rangle < 0\} \quad \text{and} \quad \{x \mid \langle g_k, x - x_k \rangle > 0\}$$

while the *cutting plane* $\{x \mid \langle g_k, x - x_k \rangle = 0\}$ passes through the center of the ellipsoid $\mathscr{E}_k$ for each k. Since $f(x) > f(x_k)$ whenever $\langle g_k, x - x_k \rangle > 0$, the optimal solution x_{opt} is guaranteed to be located in $\mathscr{H}_k$. The ellipsoid defined in Step 3 contains $\mathscr{H}_k$ and is the smallest volume ellipsoid with this property. Iterating over k, the algorithm produces a sequence of ellipsoids $\mathscr{E}_0, \mathscr{E}_1, \mathscr{E}_2, \ldots$ whose volumes decrease according to

$$\text{vol}(\mathscr{E}_{k+1}) = \det(P_{k+1}) \leq e^{-\frac{1}{2n}} \det(P_k) = e^{-\frac{1}{2n}} \text{vol}(\mathscr{E}_k)$$

and where each ellipsoid is guaranteed to contain x_{opt}. The sequence of centers $x_0, x_1, x_2, \ldots$ of the ellipsoids generate a sequence of function evaluations $f(x_k)$ which *converges to the optimal value* $f(x_{\mathrm{opt}})$. Convergence of the algorithm is in 'polynomial time' due to the fact that the volume of the ellipsoids decreases geometrically. Since $x_{\mathrm{opt}} \in \mathscr{E}_k$ for all k, we have

$$f(x_k) \geq f(x_{\mathrm{opt}}) \geq f(x_k) + \langle g_k, x_{\mathrm{opt}} - x_k \rangle \geq$$
$$\geq f(x_k) + \inf_{\xi \in \mathscr{E}_k} \langle g_k, \xi - x_k \rangle = f(x_k) - \sqrt{g_k^\top P_k g_k}$$

so that $L_k \leq f(x_{\mathrm{opt}}) \leq U_k$ define an upper and lower bound on the optimal value.

The algorithm is easy to implement, is very robust from a numerical point of view and implies low memory requirements in its performance. However, convergence may be rather slow which may be a disadvantage for large optimization problems.

1.4 Linear matrix inequalities

1.4.1 What are they?

A *linear matrix inequality* is an expression of the form

$$F(x) := F_0 + x_1 F_1 + \ldots + x_n F_n \prec 0 \tag{1.4.1}$$

where

- $x = (x_1, \ldots, x_n)$ is a vector of n real numbers called the *decision variables*.

- $F_0, \ldots, F_n$ are Hermitian matrices, i.e., $F_j = F_j^\top \in \mathbb{H}$, for $j = 0, \ldots, n$.

- the inequality $\prec 0$ in (1.4.1) means 'negative definite'. That is, $u^* F(x) u < 0$ for all non-zero complex vectors u. Because all eigenvalues of Hermitian matrices are real, (1.4.1) is equivalent to saying that all eigenvalues $\lambda(F(x))$ are negative. Equivalently, the maximal eigenvalue $\lambda_{\max}(F(x)) < 0$.

Definition 1.22 (Linear Matrix Inequality) A *linear matrix inequality* (LMI) is an inequality

$$F(x) \prec 0 \tag{1.4.2}$$

where F is an *affine function* mapping a finite dimensional vector space $\mathscr{X}$ to either the set $\mathbb{H}$ of Hermitian or the set $\mathbb{S}$ of symmetric matrices.

Remark 1.23 Recall from Definition 1.10 that an affine mapping $F : \mathscr{X} \to \mathbb{H}$ necessarily takes the form $F(x) = F_0 + T(x)$ where $F_0 \in \mathbb{H}$ (i.e., F_0 is Hermitian) and $T : \mathscr{X} \to \mathbb{H}$ is a linear transformation.

Thus if $\mathscr{X}$ is finite dimensional, say of dimension n, and $\{e_1, \ldots, e_n\}$ constitutes a basis for $\mathscr{X}$, then every $x \in \mathscr{X}$ can be represented as $x = \sum_{j=1}^{n} x_j e_j$ and we can write

$$T(x) = T\left(\sum_{j=1}^{n} x_j e_j\right) = \sum_{j=1}^{n} x_j F_j$$

where $F_j = T(e_j) \in \mathbb{H}$. Hence we obtain the standard form (1.4.1) as a special case.

Remark 1.24 In most control applications, LMI's arise as functions of *matrix variables* rather than scalar valued decision variables. This means that we consider inequalities of the form (1.4.2) where $\mathscr{X}$ is, for example, the set $\mathbb{R}^{m_1 \times m_2}$ of real matrices of dimension $m_1 \times m_2$. A simple example with $m_1 = m_2 = m$ is the Lyapunov inequality $F(X) = A^\top X + X^\top A + Q \prec 0$ where $A, Q \in \mathbb{R}^{m \times m}$ are assumed to be given and X is the unknown matrix variable of dimension $m \times m$. Note that this defines an LMI only if $Q \in \mathbb{S}^m$ (as otherwise $F(X) \notin \mathbb{S}$). We can view LMI's with matrix variables as a special case of (1.4.1) by defining an arbitrary basis $E_1, \ldots, E_n$ of $\mathscr{X}$ and expanding $X \in \mathscr{X}$ as $X = \sum_{j=1}^{n} x_j E_j$. Then

$$F(X) = F\left(\sum_{j=1}^{n} x_j E_j\right) = F_0 + \sum_{j=1}^{n} x_j F(E_j) = F_0 + \sum_{j=1}^{n} x_j F_j \prec 0$$

which is of the form (1.4.1). The coefficients x_j in the expansion of X then define the decision variables. The number of (independent) decision variables n corresponds to the dimension of $\mathscr{X}$. The number n is at most m^2 (or $m_1 \times m_2$ for non-square matrix variables) and will depend on the structure imposed on the matrix variable X. For example, if the matrix variable X is required to be symmetric, $\mathscr{X} = \mathbb{S}^m$ which has a basis of $n = m(m+1)/2$ matrix-valued elements. If X is required to be diagonal then $n = m$.

Remark 1.25 A *non-strict LMI* is a linear matrix inequality where $\prec$ in (1.4.1) and (1.4.2) is replaced by $\preccurlyeq$. The matrix inequalities $F(x) \succ 0$, and $F(x) \prec G(x)$ with F and G affine functions are obtained as special cases of Definition 1.22 as they can be rewritten as the linear matrix inequalities $-F(x) \prec 0$ and $F(x) - G(x) \prec 0$. It is for this reason that we will refer to all of these inequalities as linear matrix inequalities.

1.4.2 Why are they interesting?

The linear matrix inequality (1.4.2) defines a *convex constraint* on x. That is, the set

$$\mathscr{S} := \{x \mid F(x) \prec 0\}$$

of solutions of the LMI $F(x) \prec 0$ is convex. This is an immediate consequence of Proposition 1.8 by observing that affine functions F are convex and that $\mathscr{S}$ is, in fact, equal to the sublevel set $\mathscr{S}_0$.

Although the convex constraint $F(x) \prec 0$ on x may seem rather special, it turns out that many convex sets can be represented in this way and that these sets have more attractive properties than general

convex sets. In this subsection we discuss some seemingly trivial properties of linear matrix inequalities which turn out to be of eminent help to reduce multiple constraints on an unknown variable to an equivalent constraint involving a single linear matrix inequality.

Definition 1.26 (System of LMI's) A *system of linear matrix inequalities* is a finite set of linear matrix inequalities

$$F_1(x) \prec 0, \ \ldots, \ F_K(x) \prec 0. \tag{1.4.3}$$

From Proposition 1.4 we infer that the intersection of the feasible sets of each of the inequalities (1.4.3) is convex. In other words, it is no surprise that the set of all x that satisfy (1.4.3) is convex. The question arises whether or not this set can be represented as the feasibility set of another LMI. The answer is yes. Indeed, $F_1(x) \prec 0, \ldots, F_K(x) \prec 0$ if and only if

$$F(x) := \begin{pmatrix} F_1(x) & 0 & \ldots & 0 \\ 0 & F_2(x) & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & F_K(x) \end{pmatrix} \prec 0$$

where the last inequality indeed makes sense as $F(x)$ is symmetric (or Hermitian) for any x. Further, since the set of eigenvalues of $F(x)$ is simply the union of the eigenvalues of $F_1(x), \ldots, F_K(x)$, any x that satisfies $F(x) \prec 0$ also satisfies the system of LMI's (1.4.3) and vice versa. Conclude that multiple LMI constraints can always be converted into one single LMI constraint.

A second important property amounts to incorporating *affine constraints* in linear matrix inequalities. By this, we mean that *combined constraints* (in the unknown x) of the form

$$\begin{cases} F(x) \prec 0 \\ Ax = a \end{cases} \quad \text{or} \quad \begin{cases} F(x) \prec 0 \\ x = Bu + b \text{ for some } u \end{cases}$$

where the affine function $F : \mathbb{R}^n \to \mathbb{S}$, matrices A and B and vectors a and b are given, can be *lumped* in one linear matrix inequality $G(y) \prec 0$. More generally, the combined equations

$$\begin{cases} F(x) \prec 0 \\ x \in \mathscr{M} \end{cases} \tag{1.4.4}$$

where $\mathscr{M}$ is an *affine set* in $\mathbb{R}^n$ can be rewritten in the form of one single linear matrix inequality $G(y) \prec 0$ so as to *eliminate* the affine constraint. To do this, recall that affine sets $\mathscr{M}$ can be written as

$$\mathscr{M} = \{x \mid x = x_0 + m, \, m \in \mathscr{M}_0\}$$

with $x_0 \in \mathbb{R}^n$ and $\mathscr{M}_0$ a linear subspace of $\mathbb{R}^n$. Suppose that $n_0 = \dim(\mathscr{M}_0)$ and let $e_1, \ldots, e_{n_0} \in \mathbb{R}^n$ be a basis of $\mathscr{M}_0$. Let $F(x) = F_0 + T(x)$ be decomposed as in Remark 1.23. Then (1.4.4) can be

rewritten as

$$0 \succ F(x) = F_0 + T(x_0 + \sum_{j=1}^{n_0} x_j e_j) =$$

$$= F_0 + T(x_0) + \sum_{j=1}^{n_0} x_j T(e_j) = G_0 + x_1 G_1 + \ldots + x_{n_0} G_{n_0} =: G(y)$$

where $G_0 = F_0 + T(x_0)$, $G_j = T(e_j)$ and $y = \mathrm{col}(x_1, \ldots, x_{n_0})$ are the coefficients of $x - x_0$ in the basis of $\mathcal{M}_0$. This implies that $x \in \mathbb{R}^n$ satisfies (1.4.4) if and only if $G(y) \prec 0$. With the elimination of the affine constraint, the dimension n_0 of y is generally smaller and actually at most equal to the dimension n of x.

A third property of LMI's is obtained from a simple algebraic observation. If M is a square matrix and T is non-singular, then the product T^*MT is called a *congruence transformation* of M. For Hermitian and symmetric matrices M such a transformation does not change the number of positive and negative eigenvalues of M (See Exercise 15). Indeed, if vectors u and v are related according to $u = Tv$ with T non-singular, then $u^*Mu < 0$ for all nonzero u is equivalent to saying that $v^*T^*MTv < 0$ for all nonzero v. Hence $M \prec 0$ if and only if $T^*MT \prec 0$. Applying this insight to a partitioned Hermitian matrix

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

with M_{11} square and non-singular, we obtain for a special congruence transformation that

$$M \prec 0 \iff \begin{pmatrix} I & -M_{11}^{-1}M_{12} \\ 0 & I \end{pmatrix}^\top \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} I & -M_{11}^{-1}M_{12} \\ 0 & I \end{pmatrix} \prec 0$$

$$\iff \begin{pmatrix} M_{11} & 0 \\ 0 & S \end{pmatrix} \prec 0$$

$$\iff \begin{cases} M_{11} \prec 0 \\ S \prec 0 \end{cases}$$

where

$$S := M_{22} - M_{21}M_{11}^{-1}M_{12}$$

is the so called *Schur complement* of M_{11} in M. A similar result is obtained by computing the congruence transformation of M with

$$T = \begin{pmatrix} I & 0 \\ -M_{22}^{-1}M_{21} & I \end{pmatrix}$$

This observation can be exploited to derive a very powerful result to linearize some *non-linear inequalities* to *linear inequalities*:

Proposition 1.27 (Schur complement) *Let $F : \mathscr{X} \to \mathbb{H}$ be an affine function which is partitioned according to*

$$F(x) = \begin{pmatrix} F_{11}(x) & F_{12}(x) \\ F_{21}(x) & F_{22}(x) \end{pmatrix}$$

where $F_{11}(x)$ is square. Then $F(x) \prec 0$ if and only if

$$\begin{cases} F_{11}(x) \prec 0 \\ F_{22}(x) - F_{21}(x)\left[F_{11}(x)\right]^{-1}F_{12}(x) \prec 0. \end{cases} \quad (1.4.5)$$

if and only if

$$\begin{cases} F_{22}(x) \prec 0 \\ F_{11}(x) - F_{12}(x)\left[F_{22}(x)\right]^{-1}F_{21}(x) \prec 0. \end{cases} \quad (1.4.6)$$

The second inequalities in (1.4.5) and (1.4.6) are actually *rational* constraints in x. Using this result, it follows that non-linear matrix inequalities of the form (1.4.5) or (1.4.6) can be converted to linear matrix inequalities. In particular, the rational and non-linear inequalities of the form (1.4.5) or (1.4.6) define convex constraints on the variable x in the sense that the solution set of these inequalities is convex and can be expressed as a regular LMI.

1.4.3 What are they good for?

As we will see, many optimization problems in control, identification and signal processing can be formulated (or reformulated) using linear matrix inequalities. Clearly, it only makes sense to cast these problems in an LMI setting if these inequalities can be solved in an efficient and reliable way. Since the linear matrix inequality $F(x) \prec 0$ defines a *convex constraint* on the variable x, optimization problems involving the minimization (or maximization) of a performance function $f : \mathscr{S} \to \mathbb{R}$ with $\mathscr{S} := \{x \mid F(x) \prec 0\}$ belong to the class of *convex optimization problems*. Casting this in the setting of the previous section, it may be apparent that the full power of convex optimization theory can be employed if the performance function f is known to be convex.

Suppose that $F : \mathscr{X} \to \mathbb{S}$ is affine. There are two generic problems related to the study of linear matrix inequalities:

(a) **Feasibility:** The question whether or not there exist elements $x \in \mathscr{X}$ such that $F(x) \prec 0$ is called a *feasibility problem*. The LMI $F(x) \prec 0$ is called *feasible* if such x exists, otherwise it is said to be *infeasible*.

(b) **Optimization:** Let an *objective function* $f : \mathscr{S} \to \mathbb{R}$ where $\mathscr{S} = \{x \mid F(x) \prec 0\}$. The problem to determine

$$f_{\text{opt}} = \inf_{x \in \mathscr{S}} f(x)$$

is called an *optimization problem with an LMI constraint*. This problem involves the determination of f_{opt}, the calculation of an *almost optimal solution* x (i.e., for arbitrary $\varepsilon > 0$ the calculation of an $x \in \mathscr{S}$ such that $f_{\text{opt}} \leq f(x) \leq f_{\text{opt}} + \varepsilon$), or the calculation of optimal solutions x_{opt} (elements $x_{\text{opt}} \in \mathscr{S}$ such that $f_{\text{opt}} = f(x_{\text{opt}})$).

Let us give some simple examples to motivate the study of these problems.

Example 1: stability

Consider the problem to determine exponential stability of the linear autonomous system

$$\dot{x} = Ax \tag{1.4.7}$$

where $A \in \mathbb{R}^{n \times n}$. By this, we mean the problem to decide whether or not there exists positive constants M and $\alpha > 0$ such that for any initial condition x_0 the solution $x(t)$ of (1.4.7) with $x(t_0) = x_0$ satisfies the bound

$$\|x(t)\| \leq \|x(t_0)\| M e^{-\alpha(t-t_0)}, \quad \text{for all } t \geq t_0. \tag{1.4.8}$$

Lyapunov taught us that the system (1.4.7) is exponentially stable if and only if there exists $X = X^\top \in \mathbb{S}^n$ such that $X \succ 0$ and $A^\top X + XA \prec 0$. Indeed, in that case the function $V(x) := x^\top X x$ qualifies as a Lyapunov function in that it is positive for all non-zero x and strictly decaying along solutions x of (1.4.7). In Chapter 3 we show that (1.4.8) holds with $M^2 = \lambda_{\max}(X)/\lambda_{\min}(X)$ and $\alpha > 0$ where $A^\top X + XA + \alpha X \preccurlyeq -\alpha I$. Thus, exponential stability of the system (1.4.7) is equivalent to the feasibility of the LMI

$$\begin{pmatrix} -X & 0 \\ 0 & A^\top X + XA \end{pmatrix} \prec 0.$$

Example 2: μ-analysis

Experts in μ-analysis (but other people as well!) regularly face the problem to determine a diagonal matrix D such that $\|DMD^{-1}\| < 1$ where M is some given matrix. Since

$$\|DMD^{-1}\| < 1 \Longleftrightarrow D^{-\top} M^\top D^\top DMD^{-1} \prec I$$
$$\Longleftrightarrow M^\top D^\top DM \prec D^\top D$$
$$\Longleftrightarrow M^\top XM - X \prec 0$$

where $X := D^\top D \succ 0$, we see that the existence of such a matrix is an LMI feasibility problem where $\mathscr{X}$ needs to be taken as the set of diagonal matrices.

Example 3: singular value minimization

Let $F : \mathscr{X} \to \mathbb{S}$ be an affine function and let $\sigma_{\max}(\cdot)$ denote the maximal singular value of a matrix. Consider the problem to minimize $f(x) := \sigma_{\max}(F(x))$ over x. Clearly,

$$f(x) < \gamma \Longleftrightarrow \lambda_{\max}\left(F(x)^\top F(x)\right) < \gamma^2 \Longleftrightarrow \frac{1}{\gamma} F(x)^\top F(x) - \gamma I \prec 0$$
$$\Longleftrightarrow \begin{pmatrix} -\gamma I & F(x) \\ F(x)^\top & -\gamma I \end{pmatrix} \prec 0$$

where the latter inequality is obtained by taking a Schur complement. If we define

$$y := \begin{pmatrix} \gamma \\ x \end{pmatrix}, \quad G(y) := \begin{pmatrix} -\gamma I & F(x) \\ F(x)^\top & -\gamma I \end{pmatrix}, \quad g(y) := \gamma$$

then G is an *affine function* of y and the problem to minimize f over x is equivalent to the problem to minimize g over y subject to $G(y) \prec 0$. Hence, this is an optimization problem with an LMI constraint and a linear objective function g.

Example 4: simultaneous stabilization

Consider k linear time-invariant systems of the form

$$\dot{x} = A_i x + B_i u$$

where $A_i \in \mathbb{R}^{n \times n}$ and $B_i \in \mathbb{R}^{n \times m}$, $i = 1, \ldots, k$. The question of *simultaneous stabilization* amounts to finding a state feedback law $u = Fx$ with $F \in \mathbb{R}^{m \times n}$ such that each of the k autonomous systems $\dot{x} = (A_i + B_i F)x$, $i = 1, \ldots, k$ is asymptotically stable. Using Example 1 above, this problem is solved when we can find matrices F and X_i, $i = 1, \ldots, k$, such that for all of these i's

$$\begin{cases} X_i \succ 0 \\ (A_i + B_i F)^\top X_i + X_i (A_i + B_i F) \prec 0 \end{cases}. \tag{1.4.9}$$

Since both X_i and F are unknown, this is *not* a system of LMI's in the variables X_i and F. One way out of this inconvenience is to require that $X_1 = \ldots = X_k =: X$. After introducing new decision variables $Y = X^{-1}$ and $K = FY$ (1.4.9) reads

$$\begin{cases} Y \succ 0 \\ A_i Y + Y A_i^\top + B_i K + K^\top B_i^\top \prec 0 \end{cases}$$

for $i = 1, \ldots, k$. The latter is a system of LMI's in the variables Y and K. The joint stabilization problem therefore has a solution $F = KY^{-1}$ if the latter system of LMI's is feasible. We will see in Chapter 3 that the quadratic function $V(x) := x^\top X x$ serves as a joint Lyapunov function for the k autonomous systems.

Example 5: evaluation of quadratic costs

Consider the linear autonomous system

$$\dot{x} = Ax, \qquad x(0) = x_0 \tag{1.4.10}$$

together with the criterion function $J := \int_0^\infty x^\top(t) Q x(t) \mathrm{d}t$ where $Q = Q^\top \succcurlyeq 0$. Assume that the system is asymptotically stable. Then all solutions x of (1.4.10) are square integrable so that $J < \infty$.

Now consider the non-strict linear matrix inequalities $X \succcurlyeq 0$ and $A^\top X + XA + Q \preccurlyeq 0$. For any feasible $X = X^\top$ we can differentiate $x^\top(t)Xx(t)$ along solutions x of (1.4.10) to obtain

$$\frac{d}{dt}[x(t)^\top Xx(t)] = x(t)^\top[A^\top X + XA]x(t) \leq -x(t)^\top Qx(t).$$

Integrating the latter inequality from $t = 0$ till ∞ yields the upper bound

$$J = \int_0^\infty x^\top(t)Qx(t)\mathrm{d}t \leq x_0^\top Xx_0.$$

where we used that $\lim_{t\to\infty} x(t) = 0$. Moreover, the smallest upperbound of J is obtained by minimizing the function $f(X) := x_0^\top Xx_0$ over all $X = X^\top$ which satisfy $X \succcurlyeq 0$ and $A^\top X + XA + Q \preccurlyeq 0$. Again, this is an optimization problem with an LMI constraint.

Example 6: a Leontief economy

A manufacturer is able to produce n different products from m different resources. Assume that the selling price of product j is p_j and that it takes the manufacturer a_{ij} units of resource i to produce one unit of product j. Let x_j denote the amount of product j that is to be produced and let a_i denote the amount of available units of resource i, $i = 1, \ldots, m$. A smart manager advised the manufacturer to maximize her profit

$$p(x_1, \ldots, x_n) := p_1x_1 + p_2x_2 + \ldots + p_nx_n,$$

but the manager realized that she can do this only subject to the production constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq a_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq a_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq a_m$$

and $x_j \geq 0$, $j = 1, \ldots, n$. Note that the manufacturer faces an optimization problem subject to a system of non-strict linear matrix inequalities.

> Wassily Leontief was born in 1906 in St. Petersburg and is winner of the 1973 Nobel Prize of Economics. Among many things, he used input-output analysis to study the characteristics of trade flow between the U.S. and other countries.

1.4.4   How are they solved?

The problems defined in the previous subsection can be solved with efficient numerical tools. In this section we discuss the basic ideas behind the 'LMI-solvers'.

Ellipsoid method

We first give a solution which is based on the Ellipsoid Algorithm 1.21 discussed in Section 1.3. As mentioned before, this algorithm is simple, numerically robust and easy to implement but may be slow for larger optimization problems.

We will apply this algorithm to the feasibility problem defined in subsection 1.4.3. Let $F : \mathbb{R}^n \to \mathbb{S}$ be affine. Recall that $F(x) \prec 0$ if and only if $\lambda_{\max}(F(x)) < 0$. Define, for $x \in \mathbb{R}^n$, the function $f(x) := \lambda_{\max}(F(x))$ and consider the optimal value $V_{\text{opt}} := \inf_{x \in \mathbb{R}^n} f(x)$. Then the LMI $F(x) \prec 0$ is feasible if and only if $V_{\text{opt}} < 0$. It is infeasible if and only if $V_{\text{opt}} \geq 0$.

There are a few observations to make to apply Proposition 1.19 and the ellipsoid algorithm for this optimization problem. The first one is to establish that f is a convex function. Indeed, $\mathscr{S}$ is convex and for all $0 < \alpha < 1$ and $x_1, x_2 \in \mathscr{S}$ we have that

$$
\begin{aligned}
f(\alpha x_1 + (1-\alpha)x_2) &= \lambda_{\max}(F(\alpha x_1 + (1-\alpha)x_2)) \\
&= \lambda_{\max}(\alpha F(x_1) + (1-\alpha)F(x_2)) \\
&\leq \alpha \lambda_{\max}(F(x_1)) + (1-\alpha)\lambda_{\max}(F(x_2)) \\
&= \alpha f(x_1) + (1-\alpha)f(x_2)
\end{aligned}
$$

which shows that f is convex. Second, to apply Step 1 of the algorithm, for any x_0 we need to determine a subgradient g of f at the point x_0. To do this, we will use the fact that

$$
f(x) = \lambda_{\max}(F(x)) = \max_{u^\top u = 1} u^\top F(x) u.
$$

This means that for an arbitrary $x_0 \in \mathbb{R}^n$ we can determine a vector u_0, depending on x_0, with unit norm $u_0^\top u_0 = 1$ such that $\lambda_{\max}(F(x_0)) = u_0^\top F(x_0) u_0$. But then

$$
\begin{aligned}
f(x) - f(x_0) &= \max_{u^\top u = 1} u^\top F(x) u - u_0^\top F(x_0) u_0 \\
&\geq u_0^\top F(x) u_0 - u_0^\top F(x_0) u_0 \\
&= u_0^\top (F(x) - F(x_0)) u_0.
\end{aligned}
$$

The last expression is an affine functional which vanishes at x_0. This means that the right-hand side of this expression must be of the form $\langle g, x - x_0 \rangle$ for some vector $g \in \mathbb{R}^n$. To obtain g, we can write

$$
\begin{aligned}
u_0^\top F(x) u_0 &= \underbrace{u_0^\top F_0 u_0}_{g_0} + \sum_{j=1}^{n} x_j \underbrace{u_0^\top F_j u_0}_{g_j} \\
&= g_0 + \langle g, x \rangle,
\end{aligned}
$$

where g_j are the components of g. In particular, we obtain that $f(x) - f(x_0) \geq \langle g, x - x_0 \rangle$. The remaining steps of the ellipsoid algorithm can now be applied in a straightforward way.

Interior point methods

Optimization problems over symmetric semi-definite matrix variables belong to the realm of semi-definite programming or semi-definite optimization. In the last few decades this research field has witnessed incredible breakthroughs in numerical tools, commercial and non-commercial software developments and fast solution algorithms. In particular, the introduction of powerful interior point methods allow us to effectively decide about the feasibility of semi-definite programs and to determine their solutions. The main idea is as follows. Let F be an affine function and let $\mathscr{S} := \{x \mid F(x) \prec 0\}$ be the domain of a convex function $f : \mathscr{S} \to \mathbb{R}$ which we wish to minimize. That is, we consider the convex optimization problem

$$V_{\text{opt}} = \inf_{x \in \mathscr{S}} f(x).$$

The idea behind interior point methods is to solve this constrained optimization problem by a sequence of unconstrained optimization problems. For this purpose, a *barrier function* ϕ is introduced. This is a function $\phi : \mathbb{R}^n \to \mathbb{R}$ which is required to

(a) be *strictly convex* on the interior of $\mathscr{S}$ and

(b) approach $+\infty$ along each sequence of points $\{x_n\}_{n=1}^{\infty}$ in the interior of $\mathscr{S}$ that converges to a boundary point of $\mathscr{S}$.

Given such a barrier function ϕ, the constraint optimization problem to minimize $f(x)$ over all $x \in \mathscr{S}$ is replaced by the *unconstrained optimization problem* to minimize the functional

$$f_t(x) := f(x) + t\phi(x) \tag{1.4.11}$$

where $t > 0$ is a so called *penalty parameter*. The main idea is to determine a curve $t \mapsto x(t)$ that associates with any $t > 0$ a minimizer $x(t)$ of f_t. Subsequently, the behavior of this mapping is considered as the penalty parameter t decreases to zero. In almost all interior point methods, the unconstrained optimization problem is solved with the classical Newton-Raphson iteration technique to approximately determine a local minimizer of f_t. Since f_t is strictly convex on $\mathbb{R}^n$, every local minimizer of f_t is guaranteed to be the unique global minimizer. Under mild assumptions and for a suitably defined sequence of penalty parameters t_n, $t_n \to 0$ as $n \to \infty$, the sequence $x(t_n)$ will converge to a point x^*. That is, the limit $x^* := \lim_{n \to \infty} x(t_n)$ exists and $V_{\text{opt}} = f(x^*)$. If, in addition, x^* belongs to the interior of $\mathscr{S}$ then $V_{\text{opt}} = f(x^*)$ and x^* is an optimal solution to the original convex optimization problem; otherwise an almost optimal solution can be deduced from the sequence $x(t_n)$.

A small modification of this theme is obtained by replacing the original constraint optimization problem by the unconstrained optimization problem to minimize

$$g_t(x) := \phi_0(t - f(x)) + \phi(x) \tag{1.4.12}$$

where $t > t_0 := V_{\text{opt}}$ and ϕ_0 is a barrier function for the non-negative real half-axis. Again, the idea is to determine, for every $t > 0$ a minimizer $x(t)$ of g_t (typically using the classical Newton-Raphson algorithm) and to consider the 'path' $t \mapsto x(t)$ as function of the penalty parameter t. The

curve $t \mapsto x(t)$ with $t > t_0$ is called the *path of centers* for the optimization problem. Under suitable conditions the solutions $x(t)$ are analytic and has a limit as $t \downarrow t_0$, say x_{opt}. The point $x_{\text{opt}} := \lim_{t \downarrow t_0} x(t)$ is optimal in the sense that $V_{\text{opt}} = f(x_{\text{opt}})$ since for $t > t_0$, $x(t)$ is feasible and satisfies $f(x(t)) < t$.

Interior point methods can be applied to either of the two LMI problems defined in the previous section. If we consider the *feasibility problem* associated with the LMI $F(x) \prec 0$ then (f does not play a role and) one candidate barrier function is the logarithmic function

$$\phi(x) := \begin{cases} \log \det(-F(x)^{-1}) & \text{if } x \in \mathscr{S}, \\ \infty & \text{otherwise.} \end{cases}$$

If $\mathscr{S}$ is bounded and non-empty, ϕ will be strictly convex. By invoking Proposition 1.12, we conclude that there exists a unique x_{opt} such that $\phi(x_{\text{opt}})$ is the global minimum of ϕ. The point x_{opt} belongs to $\mathscr{S}$ and is called the *analytic center* of the feasibility set $\mathscr{S}$. It is usually obtained in a very efficient way from the classical Newton iteration

$$x_{k+1} = x_k - \left(\phi''(x_k) \right)^{-1} \phi'(x_k). \tag{1.4.13}$$

Here ϕ' and ϕ'' denote the gradient and the Hessian of ϕ, respectively. The convergence of this algorithm can be analyzed as follows. Since ϕ is strongly convex and sufficiently smooth, there exist numbers L and M such that for all vectors u with norm $\|u\| = 1$ there holds

$$u^\top \phi''(x) u \geq M$$
$$\|\phi''(x)u - \phi''(y)u\| \leq L\|x - y\|.$$

In that case,

$$\|\phi'(x_{k+1})\|^2 \leq \frac{L}{2M^2} \|\phi'(x_k)\|^2$$

so that whenever the initial value x_0 is such that $\frac{L}{2M^2}\|\phi'(x_0)\| < 1$ the method is guaranteed to converge *quadratically*.

The idea will be to implement this algorithm in such a way that quadratic convergence can be guaranteed for the largest possible set of initial values x_0. For this reason the iteration (1.4.13) is modified as follows

$$x_{k+1} = x_k - \alpha_k(\lambda(x_k)) \left(\phi''(x_k) \right)^{-1} \phi'(x_k)$$

where

$$\alpha_k(\lambda) := \begin{cases} 1 & \text{if } \lambda < 2 - \sqrt{3} \\ \frac{1}{1+\lambda} & \text{if } \lambda \geq 2 - \sqrt{3} \end{cases}.$$

and $\lambda(x) := \sqrt{\phi'(x)^\top \phi''(x) \phi'(x)}$ is the so called *Newton decrement* associated with ϕ. It is this damping factor that guarantees that x_k will converge to the analytic center x_{opt}, the unique minimizer of ϕ. It is important to note that the step-size is variable in magnitude. The algorithm guarantees that x_k is always feasible in the sense that $x_k \in \mathscr{S}$ and that x_k converges globally to a minimizer x_{opt} of ϕ. It can be shown that $\phi(x_k) - \phi(x_{\text{opt}}) \leq \varepsilon$ whenever

$$k \geq c_1 + c_2 \log\log(1/\varepsilon) + c_3 \left(\phi(x_0) - \phi(x_{\text{opt}}) \right)$$

where c_1, c_2 and c_3 are constants. The first and second terms on the right-hand side do not dependent on the optimization criterion and the specific LMI constraint. The second term can almost be neglected for small values of ε.

The *LMI optimization problem* to minimize $f(x)$ subject to the LMI $F(x) \prec 0$ can be viewed as a feasibility problem for the LMI

$$G_t(x) := \begin{pmatrix} f(x) - t & 0 \\ 0 & F(x) \end{pmatrix} \prec 0$$

where $t > t_0 := \inf_{x \in \mathscr{S}} f(x)$ is a penalty parameter. Using the same barrier function yields the unconstrained optimization problem to minimize

$$g_t(x) := \log \det(-G_t(x)^{-1}) = \log \frac{1}{t - f(x)} + \log \det(-F(x)^{-1})$$

for a sequence of decreasing positive values of t. Due to the strict convexity of g_t the minimizer $x(t)$ of g_t is unique for all $t > t_0$. Since closed form expressions for the gradient and Hessian of g_t can be obtained, a Newton iteration is an efficient numerical method to find minimizers of g_t. Currently much research is devoted to further exploiting the structure of LMI's in order to tailor dedicated solvers for specific semi-definite programs.

1.4.5 What are dual LMI problems?

Let $\mathscr{X} = \mathbb{R}^n$ be a finite dimensional vector space and consider a *linear objective function* $f : \mathscr{X} \to \mathbb{R}$ by setting $f(x) = c^\top x = \langle x, c \rangle$ where c is a vector in $\mathbb{R}^n$. As in subsection 1.3.3, consider the primal optimization problem to determine

$$P_{\text{opt}} = \inf_{x \in \mathscr{S}} f(x)$$

where $\mathscr{S}$ is defined, as in (1.3.4), by

$$\mathscr{S} = \{x \in \mathscr{X} \mid G(x) \preccurlyeq 0, H(x) = 0\}.$$

Here, $G : \mathscr{X} \to \mathbb{S}^k$ and $H : \mathscr{X} \to \mathbb{S}^l$ are affine functions. That is, $G(x) = G_0 + G_1(x)$ and $H(x) = H_0 + H_1(x)$ where $G_0 \in \mathbb{S}^k$, $H_0 \in \mathbb{S}^l$ are symmetric matrices and G_1 and H_1 are linear mappings defined on $\mathbb{R}^n$. Following the terminology of subsection 1.3.3, this is an example of a *linear convex program*. The aim of this subsection is to establish a precise formulation of the dual optimization problem and to characterize the solvability conditions of Theorem 1.16.

With reference to Remark 1.18, define the closed convex cone $\mathscr{K} := \{Y \in \mathbb{S}^k \mid Y \preccurlyeq 0\}$ and note that the inequality $G(x) \preccurlyeq 0$ is equivalent to $G(x) \in \mathscr{K}$. Obviously, every $Y \in \mathbb{S}^k$ defines a linear functional $a(\cdot) := \langle Y, \cdot \rangle$ on the set $\mathbb{S}^k$ of symmetric matrices. Conversely, every linear functional $a : \mathbb{S}^k \to \mathbb{R}$ uniquely defines an element $Y \in \mathbb{S}^k$ such that $a(\cdot) = \langle Y, \cdot \rangle$. In particular, for all $K \in \mathscr{K}$ the linear functional $a(K) = \langle Y, K \rangle$ is non-positive if and only if $Y \succcurlyeq 0$. Hence, the Lagrangian $L : \mathbb{R}^n \times \mathbb{S}^k \times \mathbb{S}^l \to \mathbb{R}$ defined by

$$L(x, Y, Z) := \langle x, c \rangle + \langle Y, G(x) \rangle + \langle Z, H(x) \rangle$$

with inner products defined on $\mathbb{R}^n$, $\mathbb{S}^k$ and $\mathbb{S}^l$ (respectively) satisfies

$$\ell(Y,Z) := \inf_{x \in \mathscr{X}} L(x,Y,Z) \leq \inf_{x \in \mathscr{S}} \langle x, c \rangle = P_{\text{opt}}.$$

for all $Y \in \mathbb{S}^k$, $Z \in \mathbb{S}^l$, $Y \succcurlyeq 0$. The dual optimization problem therefore amounts to determining

$$D_{\text{opt}} := \sup_{Y \succcurlyeq 0, Z} \ell(Y,Z) = \sup_{Y \succcurlyeq 0, Z} \inf_{x \in \mathscr{X}} L(x,Y,Z).$$

For this case, Theorem 1.16 specializes to the following result.

Theorem 1.28 *Under the conditions given in this subsection, suppose there exists* $(x_0, Y_0, Z_0) \in \mathbb{R}^n \times \mathbb{S}^k \times \mathbb{S}^l$ *such that* $G(x_0) \prec 0$, $H(x_0) = 0$, $Y_0 \succcurlyeq 0$ *and* $c + G_1^*(Y_0) + H_1^*(Z_0) = 0$. *Then both the primal and the dual optimization problem admit optimal solutions and*

$$P_{opt} = \min_{G(x) \preccurlyeq 0, H(x) = 0} \langle x, c \rangle = \max_{Y \succcurlyeq 0, c + G_1^*(Y) + H_1^*(Z) = 0} \langle G_0, Y \rangle + \langle H_0, Z \rangle = D_{opt}.$$

Moreover, the triple $(x_{opt}, Y_{opt}, Z_{opt})$ *is optimal for both the primal and the dual problem if and only if*

(a) $G(x_{opt}) \preccurlyeq 0$, $H(x_{opt}) = 0$,

(b) $Y_{opt} \succcurlyeq 0$, $c + G_1^*(Y_{opt}) + H_1^*(Z_{opt}) = 0$ *and*

(c) $\langle Y_{opt}, G(x_{opt}) \rangle = 0$.

Proof. Under the given feasibility conditions, the dual optimal value

$$\begin{aligned} D_{\text{opt}} &= \max_{Y \succcurlyeq 0, Z} \inf_x \langle c, x \rangle + \langle Y, G_0 \rangle + \langle Z, H_0 \rangle + \langle G_1^*(Y), x \rangle + \langle H_1^*(Z), x \rangle \\ &= \max_{Y \succcurlyeq 0, Z} \inf_x \langle c + G_1^*(Y) + H_1^*(Z), x \rangle + \langle G_0, Y \rangle + \langle H_0, Z \rangle \\ &= \max_{Y \succcurlyeq 0, c + G_1^*(Y) + H_1^*(Z) = 0} \langle G_0, Y \rangle + \langle H_0, Z \rangle. \end{aligned}$$

Here, G_1^* and H_1^* denote the adjoint operators of the linear mappings G_1 and H_1, respectively and the last equality follows by dualization of the dual problem. Since the dual problem satisfies the constraint qualification by assumption, the primal problem admits an optimal solution. Let $(x_{\text{opt}}, Y_{\text{opt}}, Z_{\text{opt}})$ be optimal for both the primal and the dual problem. Then items 1 and 2 are immediate and item 1 implies that $\langle x_{\text{opt}}, c \rangle \leq L(x, Y_{\text{opt}}, Z_{\text{opt}})$ for all x. With $x = x_{\text{opt}}$ this yields $\langle Y_{\text{opt}}, G(x_{\text{opt}}) \rangle \geq 0$. On the other hand, items 1 and 2 imply $\langle Y_{\text{opt}}, G(x_{\text{opt}}) \rangle \leq 0$ and we may conclude item 3. Conversely, if items 1, 2 and 3 hold, then it is easily verified that L satisfies the saddle-point property

$$L(x_{\text{opt}}, Y, Z) \leq L(x_{\text{opt}}, Y_{\text{opt}}, Z_{\text{opt}}) \leq L(x, Y_{\text{opt}}, Z_{\text{opt}}), \qquad \text{for all } x, Y \succcurlyeq 0, Z$$

The first inequality shows that $(Y_{\text{opt}}, Z_{\text{opt}})$ is an optimal solution for the dual problem. Likewise, the second inequality shows that x_{opt} is optimal for the primal problem. ∎

1.4.6   When were they invented?

Contrary to what many authors nowadays seem to suggest, the study of linear matrix inequalities in the context of dynamical systems and control goes back a long way in history and probably starts with the fundamental work of Aleksandr Mikhailovich Lyapunov on the stability of motion. Lyapunov was a school friend of Markov (yes, the one of the Markov parameters) and later a student of Chebyshev. Around 1890, Lyapunov made a systematic study of the local expansion and contraction properties of motions of dynamical systems around an attractor. He worked out the idea that an invariant set of a differential equation is stable in the sense that it attracts all solutions if one can find a function that is bounded from below and decreases along all solutions outside the invariant set.

Figure 1.2: Aleksandr Mikhailovich Lyapunov

Aleksandr Mikhailovich Lyapunov was born on May 25, 1857 and published in 1892 his work 'The General Problem of the Stability of Motion' in which he analyzed the question of stability of equilibrium motions of mechanical systems. This work served as his doctoral dissertation and was defended on September 1892 in Moscow University. Put into modern jargon, he studied stability of differential equations of the form

$$\dot{x} = A(x)$$

where $A : \mathbb{R}^n \to \mathbb{R}^n$ is some analytic function and x is a vector of positions and velocities of material taking values in a finite dimensional state space $\mathscr{X} = \mathbb{R}^n$. As Theorem I in Chapter 1, section 16 it contains the statement[4] that

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[4]Translation by A.T. Fuller as published in the special issue of the International Journal of Control in March 1992 and in [25].

if the differential equation of the disturbed motion is such that it is possible to find a definite function V of which the derivative V′ is a function of fixed sign which is opposite to that of V, or reduces identically to zero, the undisturbed motion is stable.

The intuitive idea behind this result is that the so called *Lyapunov function V* can be viewed as a generalized 'energy function' (in the context of mechanical systems the kinetic and potential energies always served as typical Lyapunov functions). A system is then stable if it is 'dissipative' in the sense that the Lyapunov function decreases. Actually, this intuitive idea turns out to be extremely fruitful in understanding the role of linear matrix inequalities in many problems related to analysis and synthesis of systems. This is why we devote the next chapter to dissipative dynamical systems. We will consider stability issues in much more detail in a later chapter.

1.5   Further reading

Foundations on convex sets and convex functions have been developed around 1900, mainly by the work of Minkowski in [30]. Detailed and classical treatments on the theory of convex functions and convex set analysis can be found in many books. We just mention the work in [1, 35, 38, 55]. For a more recent treatment we recommend the book by Boyd and Vandenberghe [4]. The theory of convex programs and its relation to Lagrange multipliers and saddle points originates with the work by Kuhn-Tucker in [23]. The constraint qualification assumption is due to Slater. For more details on the theory of subgradients we refer to [1, 39]. A standard work on optimization methods with applications in systems theory is the classical book by Luenberger [24]. Interior point methods were developed in a series of papers [20] and have led to major breakthroughs in LMI-solvers in the work of Nesterov and co-authors in [33]. For general resources on the theory of linear algebra we refer to the classical work by Golub and van Loan in [10] and Horn and Johnson in [17]. For a translation of parts of the thesis of Lyapunov we refer to [25].

1.6   Exercises

Exercise 1
Which of the following statements are true.

(a) $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \succ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(b) $A \succ B$ implies that $\lambda_{\max}(A) > \lambda_{\max}(B)$.

(c) $\lambda_{\max}(A + B) \leq \lambda_{\max}(A) + \lambda_{\max}(B)$.

(d) Write $A \in \mathbb{H}$ has $A = X + iY$ with X and Y real. Then $A \succcurlyeq 0$ if and only if $\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \succcurlyeq 0$.

(e) Let $T \in \mathbb{C}^{n \times m}$ and $A \in \mathbb{H}^n$. If $A \succcurlyeq 0$ then $T^*AT \succcurlyeq 0$.

(f) Let $T \in \mathbb{C}^{n \times m}$ and $A \in \mathbb{H}^n$. If $T^*AT \succcurlyeq 0$ then $A \succcurlyeq 0$.

Exercise 2
Give an example of a non-convex function $f : \mathscr{S} \to \mathbb{H}$ whose sublevel sets $\mathscr{S}_\Gamma$ are convex for all $\Gamma \in \mathbb{H}$.

Exercise 3
Let $f : \mathscr{S} \to \mathbb{H}$ be a convex function.

(a) Show the so called *Jensen's inequality* which states that for a convex combination $x = \sum_{i=1}^n \alpha_i x_i$ of $x_1, \ldots x_n \in \mathscr{S}$ there holds that

$$f\left(\sum_{i=1}^n \alpha_i x_i\right) \preccurlyeq \sum_{i=1}^n \alpha_i f(x_i).$$

Hint: A proof by induction on n may be the easiest.

(b) Show that $\mathrm{conv}(\mathscr{S})$ is equal to the set of all convex combinations of $\mathscr{S}$.

Exercise 4
Let $\mathscr{S}$ be a subset of a finite dimensional vector space. The *affine hull* $\mathrm{aff}(\mathscr{S})$ of $\mathscr{S}$ is the intersection of all affine sets containing $\mathscr{S}$ (cf. Definition 1.5). An *affine combination* of $x_1, \ldots, x_n \in \mathscr{S}$ is a point

$$x := \sum_{i=1}^n \alpha_i x_i$$

where $\sum_{i=1}^n \alpha_i = 1$ (cf. Definition 1.3). Show that for any set $\mathscr{S}$ in $\mathbb{R}^n$ the affine hull $\mathrm{aff}(\mathscr{S})$ is affine and consists of all affine combinations of the elements of $\mathscr{S}$. (cf. Proposition 1.6).

Exercise 5
Let $\mathscr{S}$ and $\mathscr{T}$ be finite dimensional vector spaces and let $f : \mathscr{S} \to \mathscr{T}$ be an affine function. Show that

(a) f^{-1} is affine whenever the inverse function exists.

(b) the *graph* $\{(x, y) \mid x \in \mathscr{S}, y = f(x)\}$ of f is affine.

(c) $f(\mathscr{S}')$ is affine (as a subset of $\mathscr{T}$) for every affine subset $\mathscr{S}' \subseteq \mathscr{S}$.

Exercise 6
It should not be surprising that the notion of a convex set and a convex function are related. Let $\mathscr{S} \subseteq \mathbb{R}^n$. Show that a function $f : \mathscr{S} \to \mathbb{R}$ is convex if and only if its *epigraph* $\mathscr{E}_f := \{(x, y) \mid x \in \mathscr{S}, y \in \mathbb{R}, f(x) \leq y\}$ is a convex set.

Exercise 7

Perform a feasibility test to verify the asymptotic stability of the system $\dot{x} = Ax$, where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -4 \end{pmatrix}.$$

Exercise 8

Show that

(a) the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by the quadratic form $f(x) = x^\top R x + 2 s^\top x + q$ is convex if and only if the $n \times n$ matrix $R = R^\top \succcurlyeq 0$.

(b) the intersection of the sets $\mathscr{S}_j := \{ x \in \mathbb{R}^n \mid x^\top R_j x + 2 s_j^\top x + q_j \le 0 \}$ where $j = 1, \ldots, k$ and $R_j \succcurlyeq 0$ is convex.

How does $\mathscr{S}_j$ look like if $R_j = 0$? And how if $s_j = 0$?

Exercise 9

Let $\mathscr{S}$ denote the vector space of (closed-loop) single-input and single-output rational transfer functions. Let $f_{\min} : \mathbb{R}_+ \to \mathbb{R}$ and $f_{\max} : \mathbb{R}_+ \to \mathbb{R}$ be two functions such that $f_{\min}(\cdot) \le f_{\max}(\cdot)$. Consider the following rather typical time and frequency domain specifications:

(a) Frequency response shaping: For $\gamma \ge 0$,

$$\mathscr{S}_\gamma = \{ S \in \mathscr{S} \mid \sup_{\omega \ge 0} |f_{\max}(\omega) - S(i\omega)| \le \gamma \}.$$

(b) Bandwidth shaping: For $\gamma \ge 0$,

$$\mathscr{S}_\gamma = \{ S \in \mathscr{S} \mid f_{\min}(\omega) \le |S(i\omega)| \le f_{\max}(\omega) \text{ for all } 0 \le \omega \le \frac{1}{\gamma} \}.$$

(c) Step response shaping: For $\gamma \ge 0$,

$$\mathscr{S}_\gamma = \{ S \in \mathscr{S} \mid f_{\min}(t) \le s(t) \le f_{\max}(t) \text{ for all } t \ge 0 \}.$$

Here, $s(t) = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{S(i\omega)}{i\omega} e^{i\omega t} \, d\omega$ is the step response of the system $S \in \mathscr{S}$.

(d) Over- and undershoot: For $\gamma > 0$,

$$\mathscr{S}_\gamma = \{ S \in \mathscr{S} \mid \sup_{t \ge 0} \max(s(t) - f_{\max}(t), f_{\min}(t) - s(t), 0) \le \gamma \}$$

with s the step response as defined in item 3.

Verify for each of these specifications whether or not $\mathscr{S}_\gamma$ defines a convex subset of $\mathscr{S}$.

Exercise 10
Consider the systems $\dot{x} = A_i x + B_i u$ where $i = 1, \ldots, 4$ and

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Find a state feedback law $u = Fx$ such that each of the 4 autonomous systems $\dot{x} = (A_i + B_i F)x$, $i = 1, \ldots, 4$ is asymptotically stable.

Exercise 11
In this exercise we investigate the stability of the linear time-varying system

$$\dot{x} = A(t)x \tag{1.6.1}$$

where for all $t \in \mathbb{R}_+$ the matrix $A(t)$ is a convex combination of the triple

$$A_1 := \begin{pmatrix} -1 & 1 \\ -1 & -0.2 \end{pmatrix}, \quad A_2 := \begin{pmatrix} -1 & 1 \\ -2 & -0.7 \end{pmatrix}, \quad A_3 := \begin{pmatrix} -2 & 1 \\ -1.2 & 0.4 \end{pmatrix}.$$

That is,
$$A(t) \in \text{conv}(A_1, A_2, A_3)$$

for all values of $t \in \mathbb{R}_+$. This is a *polytopic model*. It is an interesting fact that the time-varying system (1.6.1) is asymptotically stable in the sense that for any initial condition $x_0 \in \mathbb{R}^n$, the solution $x(\cdot)$ of (1.6.1) satisfies $\lim_{t \to \infty} x(t) = 0$ whenever there exists a matrix $X = X^\top \succ 0$ such that

$$A_1^\top X + X A_1 \prec 0$$
$$A_2^\top X + X A_2 \prec 0$$
$$A_3^\top X + X A_3 \prec 0.$$

(We will come to this fact later!) If such an X exists then (1.6.1) is asymptotically stable *irrespective of how fast the time variations of $A(t)$ take place*! Reformulate the question of asymptotic stability of (1.6.1) as a feasibility problem and find, if possible, a feasible solution X to this problem.

Exercise 12
Consider the dynamical system
$$\dot{x} = Ax + Bu$$

where x is an n-dimensional state and u is a scalar-valued input which is supposed to belong to $\mathscr{U} = \{u : \mathbb{R} \to \mathbb{R} \mid -1 \leq u(t) \leq 1 \text{ for all } t \geq 0\}$. Define the *null controllable subspace* of this system as the set

$$\mathscr{C} := \{x_0 \in \mathbb{R}^n \mid \exists T \geq 0 \text{ and } u \in \mathscr{U} \text{ such that } x(T) = 0\}$$

i.e., the set of initial states that can be steered to the origin of the state space in finite time with constrained inputs. Show that $\mathscr{C}$ is a convex set.

Exercise 13

Let $F : \mathcal{X} \to \mathbb{H}$ be affine and suppose that the LMI $F(x) \prec 0$ is feasible. Prove that there exists $\varepsilon > 0$ such that the LMI $F(x) + \varepsilon I \prec 0$ is also feasible. Does this statement hold for I replaced with any Hermitian matrix?

Exercise 14

Let $A, B \in \mathbb{H}^n$ and let $\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be the inner product defined by $\langle A, B \rangle := \text{trace}(AB)$. Prove, or give a counterexample of the following statements

   (a) $A \prec 0$ if and only if all its eigenvalues are negative.

   (b) $A \prec B$ implies $A \preccurlyeq B$ and $A \neq B$

   (c) $A \preccurlyeq B$ and $A \neq B$ implies $A \prec B$

   (d) $A \preccurlyeq B$ and $B \preccurlyeq C$ implies $A \preccurlyeq C$

   (e) $A \succcurlyeq 0, B \preccurlyeq 0$ implies $\langle A, B \rangle \leq 0$

Exercise 15

The *signature* of a Hermitian matrix $A \in \mathbb{H}^n$ is the triple $\text{sign}(A) = (n_-, n_0, n_+)$ with n_-, n_0, n_+ the number of negative, zero and positive eigenvalues of A, respectively. Show that $\text{sign}(A) = \text{sign}(T^*AT)$ for any nonsingular matrix $T \in \mathbb{R}^n$.

Chapter 2

Dissipative dynamical systems

2.1 Introduction

The notion of dissipativity is a most important concept in systems theory both for theoretical considerations as well as from a practical point of view. Especially in the physical sciences, dissipativity is closely related to the notion of energy. Roughly speaking, a dissipative system is characterized by the property that at any time the amount of energy which the system can conceivably supply to its environment can not exceed the amount of energy that has been supplied to it. Stated otherwise, when time evolves, a dissipative system absorbs a fraction of its supplied energy and transforms it for example into heat, an increase of entropy, mass, electro-magnetic radiation, or other kinds of energy losses. In many applications, the question whether a system is dissipative or not can be answered from physical considerations on the way the system interacts with its environment. For example, by observing that the system is an interconnection of dissipative components, or by considering systems in which a loss of energy is inherent to the behavior of the system due to friction, optical dispersion, evaporation losses, etc.

In this chapter we will formalize the notion of a dissipative dynamical system for a very general class of systems. It will be shown that linear matrix inequalities occur in a very natural way in the study of linear dissipative systems. Perhaps the most appealing setting for studying LMI's in system and control theory is within the framework of dissipative dynamical systems. It will be shown that solutions of LMI's have a natural interpretation as *storage functions* associated with a dissipative system. This interpretation will play a key role in understanding the importance of LMI's in questions related to stability, performance, robustness, and a large variety of controller design problems.

2.2   Dissipative dynamical systems

2.2.1   Definitions and examples

Consider a continuous time, time-invariant dynamical system S described by the equations

$$\dot{x} = f(x,w), \quad x(0) = x_0 \tag{2.2.1a}$$

$$z = g(x,w) \tag{2.2.1b}$$

Here, x is the state which takes its values in a *state space X*, w is the input taking its values in an *input space W* and z denotes the output of the system which assumes its values in the *output space Z*. Throughout this section, the precise representation of the system will not be relevant. What we need, though, is that for any initial condition $x_0 \in X$ and for any input w belonging to an input class $\mathscr{W}$, there exist unique and absolutely continuous signals $x : \mathbb{R}_+ \to X$ and $z : \mathbb{R}_+ \to Z$ which satisfy (2.2.1). Here, $\mathbb{R}_+ = [0,\infty)$ is the time set. In addition, the output z is assumed to depend on w in a *causal* way; that is, if $w_1 \in \mathscr{W}$ and $w_2 \in \mathscr{W}$ are two input signals that are identical on $[0,T]$ then the outputs z_1 and z_2 of (2.2.1) corresponding to the inputs w_1 and w_2 and the same (but arbitrary) initial condition $x(0) = x_0$ are also identical on $[0,T]$. The system (2.2.1) therefore generates outputs from inputs and initial conditions while future values of the inputs do not have an effect on the past outputs. Let

$$\mathsf{s} : W \times Z \to \mathbb{R}$$

be a mapping and assume that for all $t_0, t_1 \in \mathbb{R}$ and for all input-output pairs (w,z) satisfying (2.2.1) the composite function $\mathsf{s}(w(t),z(t))$ is locally absolutely integrable, i.e., $\int_{t_0}^{t_1} |\mathsf{s}(w(t),z(t))| \mathrm{d}t < \infty$. The mapping s will be referred to as the *supply function*.

Definition 2.1 (Dissipativity) The system S with supply function s is said to be *dissipative* if there exists a function $V : X \to \mathbb{R}$ such that

$$V(x(t_1)) \;\leq\; V(x(t_0)) + \int_{t_0}^{t_1} \mathsf{s}(w(t),z(t))\mathrm{d}t \tag{2.2.2}$$

for all $t_0 \leq t_1$ and all signals (w,x,z) which satisfy (2.2.1). The pair (S,s) is said to be *conservative* if equality holds in (2.2.2) for all $t_0 \leq t_1$ and all (w,x,z) satisfying (2.2.1).

Interpretation 2.2 The supply function (or *supply rate*) s should be interpreted as the supply *delivered to the system*. This means that $\mathsf{s}(w(\cdot),z(\cdot))$ represents the rate at which supply flows into the system if the system generates the input-output pair $(w(\cdot),z(\cdot))$. In other words, in the time interval $[0,T]$ work has been done *on* the system whenever $\int_0^T \mathsf{s}(w(t),z(t))\mathrm{d}t$ is positive, while work is done *by* the system if this integral is negative. The function V is called a *storage function* and generalizes the notion of an energy function for a dissipative system. With this interpretation, inequality (2.2.2) formalizes the idea that a dissipative system is characterized by the property that the change of internal storage $V(x(t_1)) - V(x(t_0))$ in any time interval $[t_0,t_1]$ will never exceed the amount of supply that flows into the system. Part of what is supplied to the system is stored, while the remaining part is dissipated. Inequality (2.2.2) will be referred to as the *dissipation inequality*.

We stress that, contrary to the definition in the classical papers [61,62], we do *not* require the storage function V in (2.2.2) to be non-negative. This difference is an important one and stems mainly from applications in mechanical and thermodynamical systems where energy or entropy functions are not necessarily bounded from below. See Example 2.3.

If the composite function $\overline{V}(t) := V(x(t))$ is differentiable, then (2.2.2) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{V}(t) = \partial_x V(x(t)) f(x(t), w(t)) \leq \mathrm{s}(w(t), z(t))$$

for all t and all solutions (w, x, z) of (2.2.1). Here, $\partial_x V$ denotes the derivative (the transpose of the gradient) of V and $\partial_x V(x(t))$ is the row vector consisting of all partial derivatives of V at the point $x(t)$. This observation makes dissipativity of a dynamical system a *local property* in the sense that (S, s) is dissipative if and only if

$$\partial_x V(x) f(x, w) \leq \mathrm{s}(w, g(x, w)) \qquad (2.2.3)$$

holds for all points $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$. In words, (2.2.3) states that the rate of change of storage along trajectories of the system will never exceed the rate of supply. We will refer to (2.2.3) as the *differential dissipation inequality*.

The classical motivation for the study of dissipativity comes from circuit theory. In the analysis of electrical networks the product of voltages and currents at the external branches of a network, i.e. the power, is an obvious supply function. Similarly, the product of forces and velocities of masses is a candidate supply function in mechanical systems. For those familiar with the theory of port-Hamiltonian systems or the theory of bond-graphs, we remark that port-Hamiltonian systems and bond-graphs can be viewed as representations of dissipative dynamical systems where inputs and outputs are taken to be effort and flow variables and the supply function is the product of these two variables. Bond-graphs and port-Hamiltonian systems are therefore special cases of dissipative systems.

Example 2.3 Consider a thermodynamic system at uniform temperature T on which mechanical work is being done at rate W and which is being heated at rate Q. Let (T, Q, W) be the external variables of such a system and assume that –either by physical or chemical principles or through experimentation– the mathematical model of the thermodynamic system has been decided upon and is given by the time invariant system (2.2.1). The first and second law of thermodynamics may then be formulated in the sense of Definition 2.1 by saying that the system S is *conservative* with respect to the supply function $\mathrm{s}_1 := (W + Q)$ and *dissipative* with respect to the supply function $\mathrm{s}_2 := -Q/T$. Indeed, the first law of thermodynamics states that for all system trajectories (T, Q, W) and all time instants $t_0 \leq t_1$,

$$E(x(t_1)) = E(x(t_0)) + \int_{t_0}^{t_1} (Q(t) + W(t)) \, \mathrm{d}t$$

(conservation of thermodynamic energy). The second law of thermodynamics states that all system trajectories satisfy

$$S(x(t_1)) \leq S(x(t_0)) + \int_{t_0}^{t_1} -\frac{Q(t)}{T(t)} \mathrm{d}t$$

for all $t_0 \leq t_1$. Here, E is called the *internal energy* and S the *entropy*. The first law promises that the change of internal energy is equal to the heat absorbed by the system and the mechanical work which is done on the system. The second law states that the entropy decreases at a higher rate than the quotient of absorbed heat and temperature. It follows that thermodynamic systems are dissipative with respect to two supply functions. Nernst's third law of thermodynamics –the entropy of any object of zero temperature is zero– is only a matter of scaling of the entropy function S and actually does not constrain the trajectories of the system.

Example 2.4 Other examples of supply functions $s : W \times Z \to \mathbb{R}$ are the quadratic forms

$$s(w,z) = w^\top z, \qquad\qquad s(w,z) = \|w\|^2 - \|z\|^2,$$
$$s(w,z) = \|w\|^2 + \|z\|^2, \qquad s(w,z) = \|w\|^2$$

which arise in network theory, bondgraph theory, scattering theory, H_∞ theory, game theory and LQ-optimal control and H_2-optimal control theory. We will come across these examples in considerable more detail later.

There are a few refinements to Definition 2.1 which are worth mentioning. Definition 2.1 can be generalized to *time-varying* systems by letting the supply rate s explicitly depend on time. Many authors have proposed a definition of dissipativity for discrete time systems, but since we can not think of any physical example of such a system, there seems little practical point in doing this. We will call a system *anti-dissipative* if the inequality (2.2.2) with $\geq$ replaced by $\leq$ holds. Another refinement consists of the idea that a system may be dissipative with respect to more than one supply function. See, for instance, Example 2.3. Also, a notion of *robust dissipativity* may be developed in which the system description (2.2.1) is not assumed to be perfectly known, but uncertain to some well defined extend. An uncertain system is then called robustly dissipative if (2.2.2) holds for all $t_0 \leq t_1$ and all trajectories (w,x,z) that can conceivably be generated by the uncertain system. See Section 5.3 in Chapter 5 for more details. The notion of *strict dissipativity* is a refinement of Definition 2.1 which will prove useful in the sequel. It is defined as follows.

Definition 2.5 (Strict dissipativity) The system S with supply function s is said to be *strictly dissipative* if there exists a storage function $V : X \to \mathbb{R}$ and an $\varepsilon > 0$ such that

$$V(x(t_1)) \ \leq \ V(x(t_0)) + \int_{t_0}^{t_1} s(w(t), z(t)) \mathrm{d}t - \varepsilon^2 \int_{t_0}^{t_1} \|w(t)\|^2 \mathrm{d}t \tag{2.2.4}$$

for all $t_0 \leq t_1$ and all trajectories (w,x,z) which satisfy (2.2.1).

A strictly dissipative system satisfies (2.2.2) with strict inequality, which justifies its name. As a final comment we mention the notion of *cyclo dissipativity* which has been introduced in [59]. For $T > 0$, the function $w : \mathbb{R} \to W$ is said to be *T-periodic* if for all $t \in \mathbb{R}$ we have that $w(t) = w(t + T)$. A system S with supply function s is called *cyclo dissipative* if for all $T > 0$ there holds

$$\int_0^T s(w(t), z(t)) \mathrm{d}t \ \geq \ 0$$

for all *T-periodic trajectories* $(w(\cdot), z(\cdot))$ which satisfy (2.2.1). Cyclo dissipativity is therefore a system property defined in terms of *T*-periodic trajectories only. The importance of this notion lies in the fact that it avoids reference to the internal state space structure of the system and requires a condition on signals in the external (input-output) behavior of the system only. It is easily seen that a dissipative system is cyclo dissipative whenever the state x is observable from (w, z), that is, whenever x is uniquely defined by any (w, z) which satisfies (2.2.1). Conversely, under some mild minimality and connectability conditions on the state X, a cyclo dissipative system is dissipative.

2.2.2 A classification of storage functions

Suppose that (S, s) is dissipative and let $x^* \in X$ be a fixed reference point in the state space of S. Instead of considering the set of all possible storage functions associated with (S, s), we will restrict attention to the set of *normalized storage functions* defined by

$$\mathscr{V}(x^*) := \{V : X \to \mathbb{R} \mid V(x^*) = 0 \text{ and } (2.2.2) \text{ holds}\}.$$

Hence, x^* is a reference point of *neutral storage*. Clearly, if V is a storage function satisfying (2.2.2), then $\widetilde{V}(x) := V(x) - V(x^*)$ also satisfies (2.2.2) and $\widetilde{V} \in \mathscr{V}(x^*)$.

Two mappings $V_{\mathrm{av}} : X \to \mathbb{R} \cup \{+\infty\}$ and $V_{\mathrm{req}} : X \to \mathbb{R} \cup \{-\infty\}$ will play a crucial role in the sequel. They are defined by

$$V_{\mathrm{av}}(x_0) := \sup\left\{-\int_0^{t_1} \mathsf{s}(w(t), z(t)) \, \mathrm{d}t \mid t_1 \geq 0; \ (w, x, z) \text{ satisfies } (2.2.1) \text{ with} \right. \tag{2.2.5a}$$
$$\left. x(0) = x_0 \text{ and } x(t_1) = x^*\right\}$$

$$V_{\mathrm{req}}(x_0) := \inf\left\{\int_{t_{-1}}^0 \mathsf{s}(w(t), z(t)) \, \mathrm{d}t \mid t_{-1} \leq 0; \ (w, x, z) \text{ satisfies } (2.2.1) \text{ with} \right. \tag{2.2.5b}$$
$$\left. x(0) = x_0 \text{ and } x(t_{-1}) = x^*\right\}$$

Here, $V_{\mathrm{av}}(x)$ denotes the maximal amount of internal storage that may be recovered from the system over all state trajectories starting in x and eventually ending in x^*. Similarly, $V_{\mathrm{req}}(x)$ reflects the minimal supply which needs to be delivered to the system in order to steer the state to x via any trajectory originating in x^*. We refer to V_{av} and V_{req} as the *available storage* and the *required supply*, (measured with respect to x^*). In (2.2.5) it is silently assumed that for $x_0 \in X$ there exists an input $w \in \mathscr{W}$ which steers the state from x^* at some time instant $t_{-1} < 0$ to x_0 at time $t = 0$ and back to x^* at time $t_1 > 0$. We call x_0 *connectable with* x^* if this property holds. If such a loop can be run in finite time for any $x_0 \in X$, then we say that *every state is connectable with* x^*. The following characterization is the main result of this section.

Proposition 2.6 *Let the system* S *be represented by* (2.2.1) *and let* s *be a supply function. Suppose that every state is connectable with* x^* *for some* $x^* \in X$. *Then the following statements are equivalent*

(a) (S, s) *is dissipative.*

(b) $-\infty < V_{av}(x) < \infty$ for all $x \in X$.

(c) $-\infty < V_{req}(x) < \infty$ for all $x \in X$.

Moreover, if one of these equivalent statements hold, then

(a) $V_{av}, V_{req} \in \mathscr{V}(x^*)$.

(b) $\{V \in \mathscr{V}(x^*)\} \Rightarrow \{$for all $x \in X$ there holds $V_{av}(x) \leq V(x) \leq V_{req}(x)\}$.

(c) $\mathscr{V}(x^*)$ is a convex set. In particular, $V_\alpha := \alpha V_{av} + (1-\alpha)V_{req} \in \mathscr{V}(x^*)$ for all $\alpha \in (0,1)$.

Interpretation 2.7 Proposition 2.6 confirms the intuitive idea that a dissipative system can neither supply nor store an infinite amount of energy during any experiment that starts or ends in a state of neutral storage. Proposition 2.6 shows that a system is dissipative if and only if the available storage and the required supply are real (finite) valued functions. Moreover, both the available storage and the required supply are possible storage functions of a dissipative system, these functions are normalized and define *extremal storage functions* in $\mathscr{V}(x^*)$ in the sense that V_{av} is the smallest and V_{req} is the largest element in $\mathscr{V}(x^*)$. In particular, for any state of a dissipative system, the available storage can not exceed its required supply. In addition, convex combinations of storage functions are storage functions.

Proof. Let (S,s) be dissipative, and let V be a storage function. Since $\widetilde{V}(x) := V(x) - V(x^*) \in \mathscr{V}(x^*)$ it follows that $\mathscr{V}(x^*) \neq \emptyset$ so that we may equally assume that $V \in \mathscr{V}(x^*)$. Let $x_0 \in X$, $t_{-1} \leq 0 \leq t_1$ and (w,x,z) satisfy (2.2.1) with $x(t_{-1}) = x(t_1) = x^*$ and $x(0) = x_0$. Since S is x^*-connectable such trajectories exist. From (2.2.2) we then infer that

$$-\infty < -\int_0^{t_1} \mathsf{s}(w(t), z(t)) \mathrm{d}t \leq \int_{t_{-1}}^0 \mathsf{s}(w(t), z(t)) \mathrm{d}t < +\infty.$$

First take in this inequality the supremum over all $t_1 \geq 0$ and $(w,x,z)|_{[0,t_1]}$ which satisfy (2.2.1) with $x(0) = x_0$ and $x(t_1) = x^*$. This yields that $-\infty < V_{av}(x_0) < \infty$. Second, by taking the infimum over all $t_{-1} \leq 0$ and $(w,x,z)|_{[t_{-1},0]}$ with $x(t_{-1}) = x^*$ and $x(0) = x_0$ we infer that $-\infty < V_{req}(x_0) < \infty$. Since x_0 is arbitrary, we obtain 2 and 3. To prove the converse implication, it suffices to show that V_{av} and V_{req} define storage functions. To see this, let $t_0 \leq t_1 \leq t_2$ and (w,x,z) satisfy (2.2.1) with $x(t_2) = x^*$. Then

$$V_{\mathrm{av}}(x(t_0)) \geq -\int_{t_0}^{t_1} \mathsf{s}(w(t), z(t)) \mathrm{d}t - \int_{t_1}^{t_2} \mathsf{s}(w(t), z(t)) \mathrm{d}t.$$

Since the second term in the right hand side of this inequality holds for arbitrary $t_2 \geq t_1$ and arbitrary $(w,x,z)|_{[t_1,t_2]}$ (with $x(t_1)$ fixed and $x(t_2) = x^*$), we can take the supremum over all such trajectories to conclude that

$$V_{\mathrm{av}}(x(t_0)) \geq -\int_{t_0}^{t_1} \mathsf{s}(w(t), z(t)) \mathrm{d}t + V_{\mathrm{av}}(x(t_1))$$

which shows that V_{av} satisfies (2.2.2). In a similar manner it is seen that V_{req} satisfies (2.2.2).

We next prove the remaining claims.

1. We already proved that V_{av} and V_{req} are storage functions. It thus remains to show that $V_{\mathrm{av}}(x^*) = V_{\mathrm{req}}(x^*) = 0$. Obviously, $V_{\mathrm{av}}(x^*) \geq 0$ and $V_{\mathrm{req}}(x^*) \leq 0$ (take $t_1 = t_{-1} = 0$ in (2.2.5)). Suppose that the latter inequalities are strict. Then, since the system is x^*-connectable, there exists $t_{-1} \leq 0 \leq t_1$ and a state trajectory x with $x(t_{-1}) = x(0) = x(t_1) = x^*$ such that $-\int_0^{t_1} \mathrm{s}(t)\mathrm{d}t > 0$ and $\int_{t_{-1}}^0 \mathrm{s}(t)\mathrm{d}t < 0$. But this contradicts the dissipation inequality (2.2.2) as both $\int_0^{t_1} \mathrm{s}(t)\mathrm{d}t \geq 0$ and $\int_{t_{-1}}^0 \mathrm{s}(t)\mathrm{d}t \geq 0$. Thus, $V_{\mathrm{av}}(x^*) = V_{\mathrm{req}}(x^*) = 0$.

2. If $V \in \mathscr{V}(x^*)$ then

$$-\int_0^{t_1} \mathrm{s}(w(t), z(t))\mathrm{d}t \leq V(x_0) \leq \int_{t_{-1}}^0 \mathrm{s}(w(t), z(t))\mathrm{d}t$$

for all $t_{-1} \leq 0 \leq t_1$ and (w, x, z) satisfying (2.2.1) with $x(t_{-1}) = x^* = x(t_1)$ and $x(0) = x_0$. Now take the supremum and infimum over all such trajectories to obtain that $V_{\mathrm{av}}(x_0) \leq V(x_0) \leq V_{\mathrm{req}}(x_0)$.

3. Follows trivially from the dissipation inequality (2.2.2). ∎

2.3 Linear dissipative systems with quadratic supply rates

In the previous section we analyzed the notion of dissipativity at a fairly high level of generality. In this section we will apply the above theory to linear input-output systems S described by

$$\dot{x} = Ax + Bw, \quad x(0) = x_0 \tag{2.3.1a}$$

$$z = Cx + Dw \tag{2.3.1b}$$

with state space $X = \mathbb{R}^n$, input space $W = \mathbb{R}^m$ and output space $Z = \mathbb{R}^p$. Let $x^* = 0$ be the point of neutral storage and consider supply functions that are general *quadratic functions* $\mathrm{s} : W \times Z \to \mathbb{R}$ defined by

$$\mathrm{s}(w, z) = \begin{pmatrix} w \\ z \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} = w^\top Q w + w^\top S z + z^\top S^\top w + z^\top R z. \tag{2.3.2}$$

Here, the matrix

$$P := \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix}$$

is a real symmetric matrix (that is, $P \in \mathbb{S}^{m+p}$) which is partitioned conform with w and z. No *a priori* definiteness assumptions are made on P.

Substituting the output equation (2.3.1b) in (2.3.2) shows that (2.3.2) can equivalently be viewed as a quadratic function in the variables x and w. Indeed,

$$\mathrm{s}(w, z) = \mathrm{s}(w, Cx + Dw) = \begin{pmatrix} x \\ w \end{pmatrix}^\top \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}$$

2.3.1 Main results

The following theorem is the main result of this chapter. It provides necessary and sufficient conditions for the pair (S, s) to be dissipative. In addition, it provides a complete parametrization of all normalized and quadratic storage functions, together with a useful frequency domain characterization of dissipativity.

Theorem 2.8 *Suppose that the system* S *described by (2.3.1) is controllable and let* s *be a quadratic supply function of the form (2.3.2). Then the following statements are equivalent.*

(a) (S, s) *is dissipative.*

(b) (S, s) *admits a quadratic storage function* $V(x) := x^\top K x$ *with* $K = K^\top \in \mathbb{S}^n$.

(c) *There exists* $K = K^\top$ *such that*

$$F(K) := \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^\top \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} - \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \preccurlyeq 0. \quad (2.3.3)$$

(d) *There exists* $K_- = K_-^\top$ *such that* $V_{av}(x) = x^\top K_- x$.

(e) *There exists* $K_+ = K_+^\top$ *such that* $V_{req}(x) = x^\top K_+ x$.

(f) *For all* $\omega \in \mathbb{R} \cup \{\infty\}$ *with* $\det(i\omega I - A) \neq 0$, *the transfer function* $T(s) := C(Is - A)^{-1}B + D$ *satisfies*

$$\begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} \succcurlyeq 0. \quad (2.3.4)$$

Moreover, if one of the above equivalent statements holds, then $V(x) := x^\top K x$ *is a quadratic storage function in* $\mathcal{V}(0)$ *if and only if* $F(K) \preccurlyeq 0$.

Proof. $(1 \Leftrightarrow 4,5)$. If (S, s) is dissipative and S is controllable then by Proposition 2.6 V_{av} and V_{req} are storage functions. We claim that both $V_{av}(x)$ and V_{req} are quadratic functions of x. This follows from [32, 60] upon noting that both V_{av} and V_{req} are defined as optimal values corresponding to an optimization problem with a quadratic cost subject to linear dynamics. Hence, if $x^* = 0$, $V_{av}(x)$ is of the form $x^\top K_- x$ and $V_{req}(x)$ takes the form $x^\top K_+ x$ for some matrices $K_- = K_-^\top$ and $K_+ = K_+^\top$. The converse implication is obvious from Proposition 2.6.

$(1 \Leftrightarrow 2)$. Using the previous argument, item 1 implies item 4. But item 4 implies item 2 by Proposition 2.6. Hence, $1 \Rightarrow 2$. The reverse implication is trivial.

$(2 \Rightarrow 3)$. If $V(x) = x^\top K x$ with $K \in \mathbb{S}^n$ is a storage function then the differential dissipation inequality (2.2.3) reads

$$2x^\top K(Ax + Bw) \leq \mathsf{s}(w, Cx + Dw)$$

for all $x \in X$ and all $w \in W$. This is equivalent to the algebraic condition

$$\begin{pmatrix} x \\ w \end{pmatrix}^\top \underbrace{\left\{ \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^\top \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} - \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \right\}}_{F(K)} \begin{pmatrix} x \\ w \end{pmatrix} \leq 0 \quad (2.3.5)$$

for all $x \in X$ and all $w \in W$. By definition, this is equivalent to saying that $F(K)$ is negative semi-definite, i.e., there exists $K \in \mathbb{S}$ such that $F(K) \preccurlyeq 0$.

$(3 \Rightarrow 2)$. If $K \in \mathbb{S}^n$ satisfies $F(K) \preccurlyeq 0$ then $(2.3.5)$ holds for all (x, w) which coincides with the differential dissipation inequality for $V(x) = x^\top K x$, i.e., (S, s) admits a quadratic storage function.

The equivalence $(1 \Leftrightarrow 6)$ is an application of Lemma 2.11, which we present below. An alternative proof for the implication $(1 \Rightarrow 6)$ can be given as follows. Let $\omega > 0$ be such that $\det(i\omega I - A) \neq 0$ and consider the (complex) harmonic input $w(t) = \exp(i\omega t) w_0$ with $w_0 \in \mathbb{R}^m$. Define $x(t) := \exp(i\omega t)(i\omega I - A)^{-1} B w_0$ and $z(t) := Cx(t) + Dw(t)$. Then $z(t) = \exp(i\omega t) T(i\omega) w_0$ and the (complex valued) triple (w, x, z) is a τ-periodic harmonic solution of $(2.3.1)$ with $\tau = 2\pi/\omega$. Moreover,

$$\mathsf{s}(w(t), z(t)) = w_0^* \begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} w_0$$

which is *constant* for all $t \in \mathbb{R}$. Now suppose that (S, s) is dissipative. Then for all $k \in \mathbb{Z}$, $x(t_0) = x(t_0 + k\tau)$ and hence $V(x(t_0)) = V(x(t_0 + k\tau))$. For $t_1 = t_0 + k\tau$, the dissipation inequality $(2.2.2)$ thus reads

$$\int_{t_0}^{t_1} \mathsf{s}(w(t), z(t)) \mathrm{d}t = \int_{t_0}^{t_1} w_0^* \begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} w_0 \, \mathrm{d}t$$

$$= k\tau w_0^* \begin{pmatrix} I \\ T(i\omega)I \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} w_0 \geq 0$$

Since $k\tau > 0$ and w_0 is arbitrary, this yields 6. ∎

We recognize in $(2.3.3)$ a non-strict linear matrix inequality. The matrix $F(K)$ is usually called the *dissipation matrix*. Observe that in the above theorem the set of *quadratic storage functions* in $\mathcal{V}(0)$ is completely characterized by the linear matrix inequality $F(K) \preccurlyeq 0$. In other words, the set of normalized quadratic storage functions associated with (S, s) coincides with the feasibility set of the system of LMI $F(K) \preccurlyeq 0$. In particular, the available storage and the required supply are quadratic storage functions and hence K_- and K_+ satisfy $F(K_-) \preccurlyeq 0$ and $F(K_+) \preccurlyeq 0$. Using Proposition 2.6, it moreover follows that any solution $K \in \mathbb{S}$ of $F(K) \preccurlyeq 0$ has the property that

$$K_- \preccurlyeq K \preccurlyeq K_+.$$

In other words, the set of symmetric solutions K of the LMI $F(K) \preccurlyeq 0$ is partially ordered and admits two extremal elements. The inequality $(2.3.4)$ is called the *frequency domain inequality*. The equivalence between statements 1 and the frequency domain characterization in statement 6 has a long history in system theory. The result goes back to Popov (1962), V.A. Yakubovich (1962)

and R. Kalman (1963) and is an application of the 'Kalman-Yakubovich-Popov Lemma', which we present and prove in subsection 2.3.2 below.

For conservative systems with quadratic supply functions a similar characterization can be given. The precise formulation is evident from Theorem 2.8 and is left to the reader. Strictly dissipative systems are characterized in a similar fashion (but under slightly different hypothesis) as follows.

Theorem 2.9 *Suppose that the system* S *is described by* (2.3.1) *where A has no eigenvalues on the imaginary axis. Let* s *be the quadratic supply function* (2.3.2). *Then the following statements are equivalent.*

(a) (S, s) *is strictly dissipative.*

(b) There exists $K = K^\top$ *such that*

$$F(K) := \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^\top \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} - \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \prec 0 \quad (2.3.6)$$

(c) The transfer function $T(s) := C(Is - A)^{-1}B + D$ *satisfies*

$$\begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} \succ 0 \quad \text{for all } \omega \in \mathbb{R} \cup \{\infty\} \quad (2.3.7)$$

Moreover, if one of the above equivalent statements holds, then $V(x) := x^\top K x$ *is a quadratic storage function satisfying* (2.2.4) *for some* $\varepsilon > 0$ *if and only if* $F(K) \prec 0$.

Proof.   (1⇒3). By definition, item 1 implies that for some $\varepsilon > 0$ the pair (S, s′) is dissipative with $s'(w, z) := s(w, z) - \varepsilon^2 \|w\|^2$. If S is controllable, Theorem 2.8 yields that for all $\omega \in \mathbb{R}$ with $\det(i\omega I - A) \neq 0$,

$$\begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \begin{pmatrix} Q - \varepsilon^2 I & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} = \begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} - \varepsilon^2 I \succcurlyeq 0 \quad (2.3.8)$$

The strict inequality (2.3.7) then follows. If S is not controllable, we use a perturbation argument to arrive at the inequality (2.3.7). Indeed, for $\delta > 0$ let B_δ be such that $B_\delta \to B$ as $\delta \to 0$ and (A, B_δ) controllable. Obviously, such B_δ exist. Define $T_\delta(s) := C(Is - A)^{-1}B_\delta + D$ and let (w, x_δ, z_δ) satisfy $\dot{x}_\delta = Ax_\delta + B_\delta w$, $z_\delta = Cx_\delta + Dw$. It follows that $B_\delta \to B$ implies that for every $w(t)$ which is bounded on the interval $[t_0, t_1]$, $x_\delta(t) \to x(t)$ and $z_\delta(t) \to z(t)$ pointwise in t as $\delta \to 0$. Here, (w, x, z) satisfy (2.3.1) and (2.2.4). Since V and s are continuous functions, the dissipation inequality (2.2.4) also holds for the perturbed system trajectories (w, x_δ, z_δ). This means that the perturbed system is strictly dissipative which, by Theorem 2.8, implies that (2.3.8) holds with $T(i\omega)$ replaced by $T_\delta(i\omega) := C(Ii\omega - A)^{-1}B_\delta + D$. Since for every $\omega \in \mathbb{R}$, $\det(Ii\omega - A) \neq 0$, we have that $T(i\omega) = \lim_{\delta \to 0} T_\delta(i\omega)$, $T(i\omega)$ satisfies (2.3.8) which, in turn, yields (2.3.7).

(3⇒2) is a consequence of Lemma 2.11, presented below.

($2 \Rightarrow 1$). If $K = K^\top$ satisfies $F(K) \prec 0$, then there exists $\varepsilon > 0$ such that

$$F'(K) := F(K) + \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon^2 I \end{pmatrix} = F(K) + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^\top \begin{pmatrix} \varepsilon^2 I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \preccurlyeq 0$$

By Theorem 2.8, this implies that (S, s') is dissipative with $s'(w,z) = s(w,z) - \varepsilon^2 \|w\|^2$. Inequality (2.2.4) therefore holds and we conclude that (S, s) is strictly dissipative. $\blacksquare$

Remark 2.10 Contrary to Theorem 2.8, the system S is not assumed to be controllable in Theorem 2.9.

The dissipation matrix $F(K)$ in (2.3.3) and (2.3.6) can be written in various equivalent and sometimes more convenient forms. Indeed:

$$F(K) = \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^\top \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} - \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} =$$

$$= \begin{pmatrix} A^\top K + KA & KB \\ B^\top K & 0 \end{pmatrix} - \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} =$$

$$= \begin{pmatrix} I & 0 \\ A & B \\ 0 & I \\ C & D \end{pmatrix}^\top \left(\begin{array}{cc|cc} 0 & K & 0 & 0 \\ K & 0 & 0 & 0 \\ \hline 0 & 0 & -Q & -S \\ 0 & 0 & -S^\top & -R \end{array} \right) \begin{pmatrix} I & 0 \\ A & B \\ 0 & I \\ C & D \end{pmatrix}$$

$$= \begin{pmatrix} A^\top K + KA - C^\top RC & KB - (SC)^\top - C^\top RD \\ B^\top K - SC - D^\top RC & -Q - SD - (SD)^\top - D^\top RD \end{pmatrix}.$$

If we set $W := Q + SD + (SD)^\top + D^\top RD$ and use a Schur complement, then the LMI $F(K) \prec 0$ is equivalent to

$$W \succ 0$$
$$A^\top K + KA - C^\top RC + \left(KB - (SC)^\top - C^\top RD \right) W^{-1} \left(B^\top K - SC - D^\top RC \right) \prec 0.$$

The latter is a quadratic inequality in the unknown K.

2.3.2   The Kalman-Yakubovich-Popov lemma

As mentioned in the proofs of Theorem 2.8 and Theorem 2.9, the Kalman-Yakubovich-Popov lemma is at the basis of the relation between frequency dependent matrix inequalities and an algebraic feasibility property of a linear matrix inequality. We will use this important result at various instances. The Lemma originates from a stability criterion of nonlinear feedback systems given by Popov in 1962 ([**?**]). Yakubovich and Kalman introduced the lemma by showing that the frequency condition of Popov is equivalent to the existence of a Lyapunov function.

We present a very general statement of the lemma which is free of any hypothesis on the system parameters. The proof which we present here is an elegant exercise in duality of semi-definite programming and largely based on the result that we stated in Theorem 1.16 of Chapter 1. The proof is inspired by [3].

Lemma 2.11 (Kalman-Yakubovich-Popov) *For any triple of matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $M \in \mathbb{S}^{(n+m) \times (n+m)} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$, the following statements are equivalent:*

(a) *There exists a symmetric matrix $K = K^\top$ such that*

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^\top \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + M \prec 0 \tag{2.3.9}$$

(b) *$M_{22} \prec 0$ and for all $\omega \in \mathbb{R}$ and complex vectors $\operatorname{col}(x, w) \neq 0$*

$$\begin{pmatrix} A - i\omega I & B \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} = 0 \quad implies \quad \begin{pmatrix} x \\ w \end{pmatrix}^* M \begin{pmatrix} x \\ w \end{pmatrix} < 0. \tag{2.3.10}$$

If (A, B) is controllable, the corresponding equivalence also holds for the non-strict inequalities.

With

$$M = - \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}$$

statement 2 of Lemma 2.11 is equivalent to the condition that for all $\omega \in \mathbb{R}$ with $\det(i\omega I - A) \neq 0$,

$$\begin{pmatrix} I \\ C(i\omega I - A)^{-1}B + D \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} I \\ C(i\omega I - A)^{-1}B + D \end{pmatrix} \prec 0.$$

Lemma 2.11 therefore reduces to the equivalence between the linear matrix inequality and the frequency domain inequality in Theorem 2.8 and Theorem 2.9. The Kalman-Yakubovich-Popov lemma therefore completes the proofs of these theorems.

Proof. (1⇒2). Let $w \in \mathbb{C}^m$ and $(i\omega I - A)x = Bw$, $\omega \in \mathbb{R}$. The implication then follows from

$$\begin{pmatrix} x \\ w \end{pmatrix}^* \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^\top \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} x \\ w \end{pmatrix}^* M \begin{pmatrix} x \\ w \end{pmatrix} = \begin{pmatrix} x \\ w \end{pmatrix}^* M \begin{pmatrix} x \\ w \end{pmatrix}.$$

(1⇐2). Suppose that (2.3.9) has *no* solution $K \in \mathbb{S}^n$. Then either $M_{22} \not\prec 0$ or $M_{22} \prec 0$. If $M_{22} \not\prec 0$ then the statement at item 2 is trivially false and the proof is complete. Suppose therefore that $M_{22} \prec 0$ and (2.3.9) is not feasible. This means that the optimal value

$$P_{\text{opt}} := \inf \left\{ \gamma \mid \exists K = K^\top \text{ such that } G(K, \gamma) := \begin{pmatrix} A^\top K + KA & KB \\ B^\top K & 0 \end{pmatrix} + M - \gamma I \preccurlyeq 0 \right\}$$

is non-negative. Note that this is a convex optimization problem with a linear objective function. As in subsection 1.4.5, let $\langle Y, X \rangle := \text{trace}(YX)$ be the natural inner product associated with the space $\mathbb{S}^{n+m}$ and infer from Theorem 1.28 of Chapter 1 that

$$\begin{aligned}
D_{\text{opt}} := \max_{Y \succcurlyeq 0} \ \inf_{K \in \mathbb{S}, \gamma \in \mathbb{R}} \ \gamma + \langle Y, G(K, \gamma) \rangle &= \\
= \max_{Y \succcurlyeq 0} \ \inf_{K \in \mathbb{S}, \gamma \in \mathbb{R}} \left\{ \langle Y, M \rangle + \langle Y, \begin{pmatrix} A & B \end{pmatrix}^\top K \begin{pmatrix} I & 0 \end{pmatrix} + \begin{pmatrix} I & 0 \end{pmatrix}^\top K \begin{pmatrix} A & B \end{pmatrix} - \gamma I \rangle \right\} &= \\
= \max \left\{ \langle Y, M \rangle \mid Y \succcurlyeq 0, \ \begin{pmatrix} A & B \end{pmatrix} Y \begin{pmatrix} I & 0 \end{pmatrix}^\top + \begin{pmatrix} I & 0 \end{pmatrix} Y \begin{pmatrix} A & B \end{pmatrix}^\top = 0 \right\} &= \\
= P_{\text{opt}} &
\end{aligned}$$

is also non-negative. Moreover, by Theorem 1.16, there exists a nonzero $Y = Y^\top$ such that

$$\langle Y, M \rangle = \text{trace}(YM) \geq 0, \quad \begin{pmatrix} A & B \end{pmatrix} Y \begin{pmatrix} I & 0 \end{pmatrix}^\top + \begin{pmatrix} I & 0 \end{pmatrix} Y \begin{pmatrix} A & B \end{pmatrix}^\top = 0, \quad Y \succcurlyeq 0. \quad (2.3.11)$$

Partition Y as the $(n+m) \times (n+m)$ matrix $Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$. We first claim that $Y_{11} \neq 0$. Indeed, for any $x \in \ker Y_{11}$, $w \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$, the inequality $Y \succcurlyeq 0$ implies

$$\text{col}(\alpha x, w)^* Y \, \text{col}(\alpha x, w) = \alpha x^\top Y_{12} w + w^\top Y_{21} \alpha x + w^\top Y_{22} w \geq 0.$$

Since α is arbitrary, this means that $x \in \ker Y_{21}$ so that $\ker Y_{11} \subseteq \ker Y_{21}$. In particular, if $Y_{11} = 0$, this implies that $Y_{21} = 0$, $Y_{12} = Y_{21}^\top = 0$ and, as $0 \leq \text{trace}(YM) = \text{trace}(Y_{22} M_{22})$ with $M_{22} \prec 0$, also $Y_{22} = 0$. This yields a contradiction as $Y \neq 0$. Hence, there exists a factorization $Y_{11} = VV^*$ with $V \in \mathbb{R}^{n \times r}$ a full rank matrix of positive rank $\text{rank} Y_{11} = \text{rank} V = r \geq 1$. A Cholesky factorization of Y is then given by

$$Y = \begin{pmatrix} V & 0 \\ W & U \end{pmatrix} \begin{pmatrix} V & 0 \\ W & U \end{pmatrix}^*$$

where $W \in \mathbb{R}^{m \times r}$ and U has rank $Y - r$ columns. The second expression in (2.3.11) now reads

$$0 = AY_{11} + BY_{21} + Y_{12}B^* + Y_{11}A^* = AVV^* + BWV^* + (AVV^* + BWV^*)^*,$$

showing that $AVV^* + BWV^*$ is a skew-symmetric matrix, i.e., $AVV^* + BWV^* = VJV^*$ where $J \in \mathbb{R}^{r \times r}$ satisfies $J + J^* = 0$. Since V has full rank this implies that also $AV + BW = VJ$. The first condition in (2.3.11) now states that

$$\begin{aligned}
0 \leq \text{trace}(YM) &= \text{trace}(\begin{pmatrix} V & 0 \\ W & U \end{pmatrix}^* \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} V & 0 \\ W & U \end{pmatrix}) \\
&= \text{trace}(\begin{pmatrix} V \\ W \end{pmatrix}^* M \begin{pmatrix} V \\ W \end{pmatrix}) + \text{trace}(U^* M_{22} U) \\
&\leq \text{trace}(\begin{pmatrix} V \\ W \end{pmatrix}^* M \begin{pmatrix} V \\ W \end{pmatrix})
\end{aligned}$$

where the last inequality follows from the fact that $M_{22} \prec 0$. Now let $J = QSQ^{-1}$ be a Schur decomposition of J. Then Q is an $r \times r$ unitary matrix ($Q^{-1} = Q^*$) and, since J is skew-symmetric, the

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Schur matrix S is a diagonal matrix with the purely imaginary eigenvalues of J on its diagonal. But then, with q_k denoting the kth column of Q,

$$0 \le \text{trace}\left(\begin{pmatrix} V \\ W \end{pmatrix}^* M \begin{pmatrix} V \\ W \end{pmatrix} \right) = \text{trace}(Q^* \begin{pmatrix} V^* & W^* \end{pmatrix} M \begin{pmatrix} V \\ W \end{pmatrix} Q) =$$

$$= \sum_{k=1}^{r} q_k^* \begin{pmatrix} V \\ W \end{pmatrix}^* M \begin{pmatrix} V \\ W \end{pmatrix} q_k$$

which shows that for at least one index $k \in [1, r]$ the corresponding term in the last summation must be non-negative. Let k be this index and define $x_k = Vq_k$, $w_k = Wq_k$ and let $i\omega_k = e_k^* S e_k$ the kth entry on the diagonal of S. Then, since V has full column rank, $\text{col}(x_k, w_k) \ne 0$. Moreover, $AV + BW = VJ$ with $J = QSQ^*$ implies that $AVQ + BWQ - VQS = 0$ whose kth column reads

$$0 = AVq_k + BWq_k - i\omega_k Vq_k = Ax_k + Bw_k - i\omega_k x_k = \begin{pmatrix} A - i\omega_k I & B \end{pmatrix} \begin{pmatrix} x_k \\ w_k \end{pmatrix}.$$

Hence, we found an $\omega_k \in \mathbb{R}$ and a non-zero vector $\text{col}(x_k, w_k)$ in the kernel of $\begin{pmatrix} A - i\omega_k I & B \end{pmatrix}$ for which

$$\begin{pmatrix} x_k \\ w_k \end{pmatrix}^* M \begin{pmatrix} x_k \\ w_k \end{pmatrix} = q_k^* \begin{pmatrix} V \\ W \end{pmatrix}^* M \begin{pmatrix} V \\ W \end{pmatrix} q_k \ge 0.$$

This falsifies statement 2 and therefore completes the proof. ∎

2.3.3 Dissipation functions and spectral factors

If the system S is dissipative with respect to the supply function s then for any storage function V, the inequality (2.2.3) implies that

$$\mathsf{d}(x, w) := \mathsf{s}(w, g(x, w)) - \partial_x V(x) f(x, w) \tag{2.3.12}$$

is non-negative for all x and w. Conversely, if there exists a non-negative function d and a differentiable $V : X \to \mathbb{R}$ for which (2.3.12) holds, then the pair (S, s) is dissipative. The function d quantifies the amount of supply that is dissipated in the system when it finds itself in state x while the input w is exerted. We will call $\mathsf{d} : X \times W \to \mathbb{R}$ a *dissipation function* for (S, s) if (2.3.12) is satisfied for a differentiable storage function $V : X \to \mathbb{R}$.

For linear systems with quadratic supply functions, (2.3.12) reads

$$\mathsf{d}(x, w) = \begin{pmatrix} x \\ w \end{pmatrix}^\top \left[\begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} - \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^\top \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} \right] \begin{pmatrix} x \\ w \end{pmatrix}$$

$$= -\begin{pmatrix} x \\ w \end{pmatrix}^\top F(K) \begin{pmatrix} x \\ w \end{pmatrix}.$$

If $K = K^\top$ is such that $F(K) \preccurlyeq 0$ (or $F(K) \prec 0$) then the dissipation matrix can be factorized as

$$-F(K) = \begin{pmatrix} M_K & N_K \end{pmatrix}^\top \begin{pmatrix} M_K & N_K \end{pmatrix}. \tag{2.3.13}$$

where $\begin{pmatrix} M_K & N_K \end{pmatrix}$ is a real partitioned matrix with $n + m$ columns and at least $r_K := \mathrm{rank}(F(K))$ rows. For any such factorization, the function

$$\mathrm{d}(x,w) := (M_K x + N_K w)^\top (M_K x + N_K w) = \|M_K x + N_K w\|^2$$

is therefore a dissipation function. If we extend the system equations (2.3.1) with the output equation $v = M_K x + N_K w$, then the output v incorporates the dissipated supply at each time instant and we infer from (2.3.12) that the extended system

$$\begin{cases} \dot{x} = Ax + Bw, & x(0) = x_0 \\ z = Cx + Dw \\ v = M_K x + N_K w \end{cases} \tag{2.3.14}$$

becomes *conservative* with respect to the quadratic supply function $\mathrm{s}'(w,z,v) := \mathrm{s}(w,z) - v^\top v$.

This observation leads to an interesting connection between dissipation functions and spectral factorizations of rational functions. A complex valued rational function $\Phi : \mathbb{C} \to \mathbb{H}$ is called a *spectral density* if $\Phi(s) = \Phi^*(s)$ and Φ is analytic on the imaginary axis. A rational function V is a *spectral factor* of Φ if $\Phi(s) = V^*(s)V(s)$ for all but finitely many $s \in \mathbb{C}$.

Theorem 2.12 *Consider the spectral density*

$$\Phi(s) = \begin{pmatrix} I \\ T(s) \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} I \\ T(s) \end{pmatrix}$$

where $T(s) = C(Is - A)^{-1}B + D$, A has no eigenvalues on the imaginary axis and where (A,B) is controllable. Then there exists a spectral factor of Φ if and only if there exists $K = K^\top$ such that $F(K) \preccurlyeq 0$. In that case, $V(s) := M_K(Is - A)^{-1}B + N_K$ is a spectral factor of Φ where M_K and N_K are defined by the factorization (2.3.13).

Proof. If $F(K) \preccurlyeq 0$ then $F(K)$ can be factorized as (2.3.13) and for any such factorization the system (2.3.14) is conservative with respect to the supply function $\mathrm{s}'(w,z,v) := \mathrm{s}(w,z) - v^\top v$. Applying Theorem 2.8 for conservative systems, this means that

$$\begin{pmatrix} I \\ T(i\omega) \\ V(i\omega) \end{pmatrix}^* \begin{pmatrix} Q & S & 0 \\ S^\top & R & 0 \\ 0 & 0 & -I \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \\ V(i\omega) \end{pmatrix} = \Phi(i\omega) - V^*(i\omega)V(i\omega) = 0$$

for all $\omega \in \mathbb{R}$. But a rational function that vanishes identically on the imaginary axis, vanishes for all $s \in \mathbb{C}$. Hence, we infer that $\Phi(s) = V^*(s)V(s)$ for all but finitely many $s \in \mathbb{C}$. Conversely, if no $K = K^\top$ exists with $F(K) \preccurlyeq 0$, it follows from Theorem 2.8 that $\Phi(i\omega) \not\succcurlyeq 0$ and hence Φ admits no factorization on $\mathbb{C}^0$. ■

2.3.4   The positive real lemma

Consider the system (2.3.1) together with the quadratic supply function

$$s(w,z) = z^\top w + w^\top z.$$

Then the following result is worth mentioning as a special case of Theorem 2.8.

Corollary 2.13 *Suppose that the system* S *described by* (2.3.1) *is controllable and has transfer function* T. *Let* $s(w,z) = z^\top w + w^\top z$ *be a supply function. Then equivalent statements are*

(a) (S, s) *is dissipative.*

(b) *the LMI*

$$\begin{pmatrix} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{pmatrix}^\top \left(\begin{array}{cc|cc} 0 & K & 0 & 0 \\ K & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -I \\ 0 & 0 & -I & 0 \end{array} \right) \begin{pmatrix} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{pmatrix} \preccurlyeq 0$$

   is feasible

(c) *For all* $\omega \in \mathbb{R} \cup \{\infty\}$ *with* $\det(i\omega I - A) \neq 0$ *one has* $T(i\omega)^* + T(i\omega) \succcurlyeq 0.$

Moreover, $V(x) = x^\top K x$ *defines a quadratic storage function if and only if* K *satisfies the above LMI.*

Corollary 2.13 is known as the *positive real lemma* and has played a crucial role in questions related to the stability of control systems and the synthesis of passive electrical networks. Transfer functions which satisfy the third statement are generally called *positive real*. Note that for single-input and single-output transfer function, positive realness is graphically verified by the condition that the Nyquist plot of the system lies entirely in the right-half complex plane.

2.3.5   The bounded real lemma

Consider the quadratic supply function

$$s(w,z) = \gamma^2 w^\top w - z^\top z \tag{2.3.15}$$

where $\gamma \geq 0$. We obtain the following result as an immediate consequence of Theorem 2.8.

Corollary 2.14 *Suppose that the system* S *described by* (2.3.1) *is controllable and has transfer function* T. *Let* $s(w,z) = \gamma^2 w^\top w - z^\top z$ *be a supply function where* $\gamma \geq 0$. *Then equivalent statements are*

(a) (S, s) *is dissipative.*

(b) The LMI

$$
\begin{pmatrix} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{pmatrix}^{\top}
\left(\begin{array}{cc|cc} 0 & K & 0 & 0 \\ K & 0 & 0 & 0 \\ \hline 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & I \end{array} \right)
\begin{pmatrix} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{pmatrix} \preccurlyeq 0
$$

is feasible.

(c) For all $\omega \in \mathbb{R}$ with $\det(i\omega I - A) \neq 0$ one has $T(i\omega)^* T(i\omega) \preccurlyeq \gamma^2 I$.

Moreover, $V(x) = x^{\top} K x$ defines a quadratic storage function if and only if K satisfies the above LMI.

Let us analyze the importance of this result. If the transfer function T of a system satisfies item 3 of Corollary 2.14 then for all frequencies $\omega \in \mathbb{R}$ for which $i\omega$ is not an eigenvalue of A and all complex vectors $\hat{w}(\omega) \in \mathbb{C}^m$ we have

$$
\hat{w}(\omega)^* T(i\omega)^* T(i\omega) \hat{w}(\omega) \leq \gamma^2 \|\hat{w}(\omega)\|^2.
$$

Now suppose that $\hat{w}$, viewed as a function of $\omega \in \mathbb{R}$ is square integrable in the sense that

$$
\|\hat{w}\|_2^2 := \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{w}(\omega)^* \hat{w}(\omega) \mathrm{d}\omega \; < \; \infty.
$$

Using Parseval's theorem, $\hat{w}$ is the Fourier transform of a function $w : \mathbb{R} \to W$ in that $\hat{w}(\omega) = \int_{-\infty}^{+\infty} w(t) e^{-i\omega t} \mathrm{d}t$ and we have that $\|\hat{w}\|_2 = \|w\|_2$ provided that the latter norm is defined as

$$
\|w\|_2^2 = \int_{-\infty}^{\infty} w(t)^{\top} w(t) \mathrm{d}t
$$

Let $\hat{z}(\omega) = T(i\omega)\hat{w}(\omega)$. Then, similarly, $\hat{z}$ is the Fourier transform of a function z that happens to satisfy (2.3.1) where x is a square integrable function (i.e., $\|x\|_2 < \infty$) defined as the inverse Fourier transform of $\hat{x}(\omega) := (i\omega I - A)^{-1} B \hat{w}(\omega)$. Consequently, item 3 is equivalent to saying that

$$
\|z\|_2^2 \leq \gamma^2 \|w\|_2
$$

for all inputs w for which $\|w\|_2 < \infty$. That is, the 2-norm of the output of (2.3.1) is uniformly bounded by γ^2 times the 2-norm of the input. When restricted to stable systems and signals (w,x,z) defined on the non-negative time axis $\mathbb{R}_+$, this crucial observation is at the basis of H_∞ optimal control theory as will be further exploited in the next chapter.

2.4 Interconnections of dissipative systems

In this section we consider the question whether the interconnection of a number of dissipative systems is again dissipative. To answer this question, we restrict attention to the case where two

dynamical systems are interconnected. Generalizations to interconnections of more than two dynamical systems follow immediately from these ideas and are left to the reader. Consider therefore two dynamical systems in state space form

$$\mathsf{S}_1 : \begin{cases} \dot{x}_1 = f_1(x_1, w_1, u_1) \\ z_1 = g_1(x_1, w_1, u_1) \\ y_1 = h_1(x_1, w_1, u_1) \end{cases} , \qquad \mathsf{S}_2 : \begin{cases} \dot{x}_2 = f_2(x_2, w_2, u_2) \\ z_2 = g_2(x_2, w_2, u_2) \\ y_2 = h_2(x_2, w_2, u_2) \end{cases} . \qquad (2.4.1)$$

with the decomposed inputs (w_1, u_1) and (w_2, u_2), respectively, and with outputs decomposed as (z_1, y_1) and (z_2, y_2) respectively. We will first need to formalize what we mean by an interconnection of these systems. The idea will be to distinguish a number of common terminals among the input and output variables of either of the two systems S_1 and S_2 and to declare an algebraic constraint (the interconnection constraint) on these variables. Here, we will assume that these common terminals are labeled with u and y. In more rigorous terms: the input and the output space of system S_i is assumed to be a Cartesian set product $W_i \times U_i$ and $Z_i \times Y_i$, respectively. We will think of the variables (w_i, z_i) as the *external variables* and as the variables (u_i, y_i) as the *interconnection or internal variables* of the interconnected system. That is, the variables (u_i, y_i), $i = 1, 2$, will serve to interconnect the two systems. Assume that
$$\dim Y_1 = \dim U_2 \quad \text{and} \quad \dim U_1 = \dim Y_2.$$

The *interconnection constraint* is then defined by the algebraic relations

$$y_1 = u_2, \qquad u_1 = y_2 \qquad (2.4.2)$$

and the interconnected system is defined by the laws (2.4.1) of S_1 and S_2 combined with the interconnection constraint (2.4.2). The idea behind this concept is visualized in Figure 2.1.
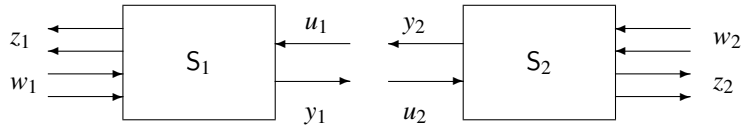

Figure 2.1: System interconnections

It is not evident that the joint equations (2.4.1)-(2.4.2) will have a unique solution for every pair of input variables (w_1, w_2) and for any pair of initial conditions in (2.4.1). Stated differently, it is not immediate that (w_1, w_2) serve as inputs and (z_1, z_2) as outputs of the interconnected system. When it does, this property is referred to as *well posedness* of the interconnection. We decided to heavily disappoint the reader and avoid a thorough discussion on this issue here. For the time being, we will assume that the interconnected system is well defined, takes (w_1, w_2) as its input variable and (z_1, z_2) as its output. Whenever well defined, the interconnected system will be denoted by $\mathsf{S} = \mathsf{S}_1 \sqcap \mathsf{S}_2$.

We now get to discuss the question whether or not the interconnection is dissipative. Suppose that both $(\mathsf{S}_1, \mathsf{s}_1)$ and $(\mathsf{S}_2, \mathsf{s}_2)$ are dissipative. Assume that the supply function s_i of the *i*th system admits

an additive structure in the sense that there exists functions $s_{i,ext} : W_i \times Z_i \to \mathbb{R}$ and $s_{i,int} : U_i \times Y_i \to \mathbb{R}$ such that

$$s_i(w_i, u_i, z_i, y_i) = s_{i,ext}(w_i, z_i) + s_{i,int}(u_i, y_i)$$

for all $(w_i, u_i, z_i, y_i) \in W_i \times U_i \times Z_i \times Y_i$. Now, consider the interconnection $\mathsf{S} = \mathsf{S}_1 \sqcap \mathsf{S}_2$ of S_1 and S_2. This interconnection is said to be *neutral* if

$$s_{1,int}(u_1, y_1) + s_{2,int}(u_2, y_2) = 0 \quad \text{for all } (u_1, u_2, y_1, y_2) \in U_1 \times U_2 \times Y_1 \times Y_2.$$

In words, this means that there is no dissipation in the interconnection variables. In physical terms: all power that flows out of system 1 through its ports that interconnect with system 2, flows into system 2 and vice versa. In mathematical terms: the interconnected system is conservative with respect to the supply function $s : U_1 \times U_2 \to \mathbb{R}$ defined as $s(u_1, u_2) := s_{1,int}(u_1, u_2) + s_{2,int}(u_2, u_1)$. Neutrality therefore seems a rather natural requirement for many interconnected systems. The following result confirms our intuition that a neutral interconnection of dissipative systems is dissipative again. It moreover shows that one candidate storage function of the interconnected system is simply the sum of the storage functions of the interconnectants.

Theorem 2.15 *Let* (S_i, s_i), $i = 1, 2$, *be dissipative dynamical systems and suppose that the interconnection* $\mathsf{S} = \mathsf{S}_1 \sqcap \mathsf{S}_2$ *is well defined and neutral. Then* S *is dissipative with respect to the supply function* $s : W_1 \times Z_1 \times W_2 \times Z_2$ *defined as*

$$s(w_1, z_1, w_2, z_2) := s_{1,ext}(w_1, z_1) + s_{2,ext}(w_2, z_2).$$

Moreover, if $V_i : X_i \to \mathbb{R}$ *is a storage function of* (S_i, s_i) *then* $V : X_1 \times X_2 \to \mathbb{R}$ *defined by* $V(x_1, x_2) := V_1(x_1) + V_2(x_2)$ *is a storage function of the interconnection* (S, s).

Proof. Since (S_i, s_i), $i = 1, 2$ is dissipative, there exists $V_1 : X_1 \to \mathbb{R}$, $V_2 : X_2 \to \mathbb{R}$ such that for all $t_0 \leq t_1$,

$$V_1(x_1(t_0)) + \int_{t_0}^{t_1} s_{1,ext}(w_1, z_1) + s_{1,int}(u_1, y_1) \mathrm{d}t \geq V_1(x_1(t_1))$$

$$V_2(x_2(t_0)) + \int_{t_0}^{t_1} s_{2,ext}(w_2, z_2) + s_{2,int}(u_2, y_2) \mathrm{d}t \geq V_2(x_2(t_1))$$

Adding these two equations and applying the neutrality condition of the interconnection yields that

$$V(x(t_0) + \int_{t_0}^{t_1} s_{1,ext}(w_1, z_1) + s_{2,ext}(w_2, z_2) \mathrm{d}t \geq V(x(t_1))$$

i.e., (S, s) is dissipative with V as its storage function. ∎

Theorem 2.15 and its proof are rather intuitive, but the result is at the basis of many things that we will have to say about robust stability of uncertain systems in later chapters.

An interesting special case of Theorem 2.15 amounts to considering *full interconnections*. In a full interconnection, all variables are used as interconnection (or internal) variables and no external

variables are left in the interconnected system. That is, for $i = 1, 2$ we have that $\dim(W_i) = \dim(Z_i) = 0$. and the supply function $s_i = s_{i,\text{int}}$ for $i = 1, 2$. Theorem 2.15 then states that a *full* interconnection of the dissipative systems (S_1, s_1) and (S_2, s_2) with $s_1 = -s_2$ yields a neutral interconnection $S = S_1 \sqcap S_2$ that is dissipative with respect to the supply function $s = 0$. With $x = \text{col}(x_1, x_2)$ denoting the state variable of S, this means that the storage function $V(x) := V(x_1, x_2) = V_1(x_1) + V_2(x_2)$ satisfies

$$V(x(t)) \leq V(x(t_0)) \quad \text{for all } t \geq t_0$$

i.e., there exists a non-increasing storage function in a full interconnection of dissipative systems. We will heavily exploit this property in investigating the stability and the robust stability of dynamical systems. For the time being, we summarize this discussion in the following corollary.

Corollary 2.16 *Let* $S = S_1 \sqcap S_2$ *be a well defined* full *interconnection of the dissipative dynamical systems* (S_1, s) *and* $(S_2, -s)$. *Then the interconnection is neutral and there exists a function V defined on the state x of* S *with the property that for all system trajectories*

$$V(x(t)) \leq V(x(t_0)) \quad \text{for all } t \geq t_0.$$

We finally remark that the above ideas can be easily generalized to interconnections of any finite number of dynamical systems.

2.5   Further reading

Many of the ideas on dissipative dynamical systems originate from the work of Willems in [61, 62]. The definitions on dissipative dynamical systems and their characterizations have been reported in [58, 59] and originate from the behavioral approach of dynamical systems. Details on the notion of cyclo-dissipativity can be found in [59] See also similar work in [54]. Extensions to nonlinear systems are discussed in [8]. A thorough system theoretic treatment of thermodynamical systems has been published in [12]. The result that linear systems that are dissipative with respect to quadratic supply functions admit quadratic storage functions is a special case of a result by Molinari who gave a rigorous proof in [32]. See also [31]. Many proofs of the Kalman-Yakubovich-Popov lemma can be found in the literature. The very general version that is presented in this chapter is very much based on a semidefinite programming duality argument that was first presented in [56] and worked out in its full generality on the basis of theorems of alternatives in [3]. For alternative proofs of this important lemma, see [6, 36, 37, 60]. For specific applications of the KYP lemma in semi-definite programming see, e.g., [5, 6, 36, 57].

2.6   Exercises

Exercise 1
Show that for conservative controllable systems the set of normalized storage functions $\mathscr{V}(x^*)$ con-

sist of one element only.
Conclude that storage functions of conservative systems are unique up to normalization!.

Exercise 2
Show that the set of dissipation functions associated with a dissipative system is convex.

Exercise 3
Consider the suspension system S of one wheel of a transport vehicle as depicted in Figure 2.2. The
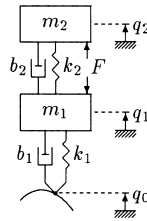

Figure 2.2: Model for suspension system

system is modeled by the equations

$$m_2\ddot{q}_2 + b_2(\dot{q}_2 - \dot{q}_1) + k_2(q_2 - q_1) - f = 0$$
$$m_1\ddot{q}_1 + b_2(\dot{q}_1 - \dot{q}_2) + k_2(q_1 - q_2) + k_1(q_1 - q_0) + f = 0$$

where f (resp. $-f$) is a force acting on the chassis mass m_2 (the axle mass m_1). Here, $q_2 - q_1$ is the distance between chassis and axle, and $\ddot{q}_2$ denotes the acceleration of the chassis mass m_2. b_2 is a damping coefficient and k_1 and k_2 are tire and air spring coefficients. ($b_1 = 0$). The variable q_0

m_1 [kg]	m_2 [kg]	k_1 [N/m]	k_2 [N/m]	b_2 [Ns/m]
5.0×10^1	4.0×10^2	3.1×10^5	3.0×10^4	1.45×10^3

Table 2.1: Physical parameters

represents the road profile. The physical parameters for one wheel of an average economy car are given in Table 2.1.

(a) Derive a state space model of the form 2.3.1 of the system which assumes $w = \mathrm{col}(q_0, F)$ and $z = \mathrm{col}(q_1, \dot{q}_1, q_2, \dot{q}_2)$ as its input and output, respectively.

(b) Define a supply function $s : W \times Z \to \mathbb{R}$ such that (S, s) is dissipative. (Base your definition on physical insight).

(c) Characterize the set of all quadratic storage functions of the system as the feasibility set of a linear matrix inequality.

(d) Compute a quadratic storage function $V(x) = x^\top K x$ for this system.

(e) Determine a dissipation function $d : X \times W \to \mathbb{R}$ for this system.

Exercise 4

Consider the transfer functions

(a) $T_1(s) = 1/(s+1)$

(b) $T_2(s) = (s-1)/(s+1)$

(c) $T_3(s) = \begin{pmatrix} (s+2)(s-1)/(s+1)^2 & (s+3)/(s+4) \\ (s-1)/(s+0.5) & (s+1)/(s+2) \end{pmatrix}$.

Determine for each of these transfer functions

(a) whether or not they are positive real and

(b) the smallest value of $\gamma > 0$ for which $T^*(i\omega)T(i\omega) \preccurlyeq \gamma^2 I$.

Reformulate these problems as a feasibility test involving a suitably defined LMI. (See Corollaries 2.13 and 2.14).

Exercise 5

Consider the electrical circuit of Figure 2.3.
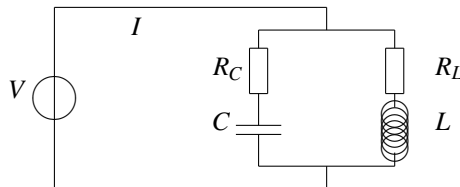
Figure 2.3: An electrical circuit

We will be interested in modeling the relation between the external voltage V and the current I through the circuit. Assume that the resistors $R_C = 1$ and $R_L = 1$, the capacitor $C = 2$ and the inductance $L = 1$.

(a) Derive a linear, time-invariant system S that models the relation between the voltage V and the current I.

(b) Find a state space representation of the form (2.3.1) which represents S. Is the choice of input and output variable unique?

(c) Define a supply function $s : W \times Z \to \mathbb{R}$ such that (S, s) is dissipative.

(d) Characterize the set of all quadratic storage functions of the system as the feasibility set of a linear matrix inequality.

(e) Compute a quadratic storage function $V(x) = x^\top K x$ of this system.

(f) Does dissipativity of (S, s) depend on whether the voltage V or the current I is taken as input of your system?

Exercise 6

Consider a first-order unstable system $P(s) = 1/(-3s + 1)$. It is desirable to design a feedback compensator C, so that the feedback system is dissipative. Assume that the compensator C is a simple gain $C(s) = k$, $k \in \mathbb{R}$. Find the range of gains k that will make the system depicted in Figure 2.4 dissipative with respect to the supply function $s(w, z) = wz$.
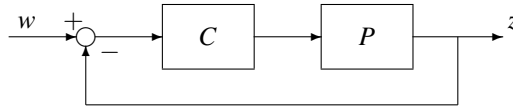

Figure 2.4: Feedback configuration

Exercise 7

The solar system is the perfect example of an autonomous system that is governed by Newton's inverse-square law. It states that the gravitational attractive force between two masses is proportional to the product of their masses and inverse proportional to the square of their distance. The orbit of a mass m moving under the influence of a Newtonian gravitational field of a second mass m is described by the inverse-square law:

$$m\ddot{z} + \frac{mMG}{\|z\|^3} z = 0. \tag{2.6.1}$$

Here, G denotes the constant of universal gravitation, $z(t) \in \mathbb{R}^3$ denotes the position of mass m at time t, where we assume that M is located in the origin. Let $p = m\dot{z}$ denote the momentum of mass m and let $x = \mathrm{col}(z, p)$ be a state vector.

(a) Represent this (autonomous) system in the form $\dot{x} = f(x)$, $z = g(x)$.

(b) Consider the function
$$V(z, p) := \frac{1}{2m} \|p\|^2 - \frac{mMG}{\|z\|}$$
and show that V is a storage function of the system with supply function $s(z) = 0$.

(c) Prove that the orbit of the mass m is a hyperbola, parabola or ellipse depending on whether $V > 0$, $V = 0$ or $V < 0$ along solutions z of (2.6.1).

Conclude from the last item that Kepler's first law of planetary motions tells us that the solar system can be viewed an autonomous dissipative system with a negative storage function.

Exercise 8

A *Hamiltonian system* is a nonlinear dynamical system of the form

$$\mathsf{S}: \quad \begin{cases} \dot{x} = (J(x) - R(x))\nabla H(x) + B(x)w \\ z = B(x)^\top \nabla H(x) \end{cases} \tag{2.6.2}$$

where $H : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function, $B : \mathbb{R}^n \to \mathbb{R}^{n \times m}$, $\nabla H(x)$ denotes the gradient vector of H at x and where, for every $x \in \mathbb{R}^n$, $J(x)$ is a real skew-symmetric matrix in that $J(x) + J(x)^\top = 0$ and $R(x) \in \mathbb{S}^n$ is positive semi-definite.

(a) Prove that a Hamiltonian system is conservative with respect to the supply function $\mathsf{s}(w, z) = z^\top w$ if and only if $R(x) = 0$ for all $x \in \mathbb{R}^n$.

(b) Prove that every Hamiltonian system is dissipative with respect to the supply function $\mathsf{s}(w, z) = z^\top w$.

(c) Prove that every linear time-invariant system (2.3.1) with $D = 0$ that is dissipative with respect to the supply function $\mathsf{s}(w, z) = z^\top w$ can be written as a Hamiltonian system (2.6.2).

Exercise 9

Is it true that every first order single-input single-output stable system with input w and output z is dissipative with respect to the supply function $\mathsf{s}(w, z) = wz$?

Chapter 3

Nominal stability and nominal performance

3.1 Lyapunov stability

As mentioned in Chapter 1, Aleksandr Mikhailovich Lyapunov studied contraction and expansion phenomena of the motions of a mechanical system around an equilibrium. Translated in modern jargon, the study of *Lyapunov stability* concerns the asymptotic behavior of the state of an autonomous dynamical system. The main contribution of Lyapunov has been to define the concept of stability, asymptotic stability and instability of such systems and to give a method of verification of these concepts in terms of the existence of functions, called Lyapunov functions. Both his definition and his verification method characterize, in a local way, the stability properties of an autonomous dynamical system. Unfortunately, for the general class of nonlinear systems there are no systematic procedures for finding Lyapunov functions. However, we will see that for linear systems the problem of finding Lyapunov functions can be solved adequately as a feasibility test of a linear matrix inequality.

3.1.1 Nominal stability of nonlinear systems

Let X be a set and $T \subseteq \mathbb{R}$. A *flow* is a mapping $\phi : T \times T \times X \to X$ which satisfies the

(a) consistency property: $\phi(t,t,x) = x$ for all $t \in T$ and $x \in X$ and the

(b) semi-group property: $\phi(t_1,t_{-1},x_0) = \phi(t_1,t_0,\phi(t_0,t_{-1},x_0))$ for all $t_{-1} \le t_0 \le t_1$ and $x_0 \in X$.

The set X is called the *state space* (or the *phase space*) and we will think of a flow as a *state evolution map*. A flow defines an *unforced* or *autonomous dynamical system* in the sense that the evolution of a flow is completely determined by an initial state and not by any kind of external input. A typical example of a flow is the solution of a differential equation of the form

$$\dot{x}(t) = f(x(t), t) \tag{3.1.1}$$

with finite dimensional state space $X = \mathbb{R}^n$ and where $f : X \times T \to X$ is a function. By a *solution* of (3.1.1) over a (finite) time interval $T \subset \mathbb{R}$ we will mean a function $x : T \to X$ which is differentiable everywhere and which satisfies (3.1.1) for all $t \in T$. By a solution over the unbounded intervals $\mathbb{R}_+$, $\mathbb{R}_-$ or $\mathbb{R}$ we will mean a solution over all finite intervals T contained in either of these sets. Let us denote by $\phi(t, t_0, x_0)$ the solution of (3.1.1) at time t with initial condition $x(t_0) = x_0$. If for all pairs $(t_0, x_0) \in T \times X$ one can guarantee existence and uniqueness of a solution $\phi(t, t_0, x_0)$ of (3.1.1) with $t \in T$, then the mapping ϕ satisfies the consistency and semi-group property, and therefore defines a flow. In that case, ϕ is said to be the *flow associated with the differential equation* (3.1.1). This flow is called *time-invariant* if

$$\phi(t + \tau, t_0 + \tau, x_0) = \phi(t, t_0, x_0)$$

for any $\tau \in T$. It is said to be *linear* if X is a vector space and $\phi(t, t_0, \cdot)$ is a linear mapping for all time instances t and t_0. For time-invariant flows we usually take (without loss of generality) $t_0 = 0$ as initial time. Existence and uniqueness of solutions of (3.1.1) is guaranteed whenever f satisfies a *global Lipschitz condition*. That is, if X has the structure of a normed vector space with norm $\|\cdot\|$, and if for some $L > 0$, the inequality

$$\|f(x, t) - f(y, t)\| \leq L \|x - y\|$$

holds for all $x, y \in X$ and $t \in T$.

An element $x^* \in X$ is a *fixed point* or an *equilibrium point* of (3.1.1) if $f(x^*, t) = 0$ for all $t \in T$. It is easy to see that x^* is a fixed point if and only if $\phi(t, t_0, x^*) = x^*$ is a solution of (3.1.1) for all t and t_0. In other words, fixed points define solutions of (3.1.1) that remain in x^* once they started there –this explains the name.

There exists a wealth of concepts to define the stability of a flow ϕ. The various notions of Lyapunov stability pertain to a distinguished fixed point x^* of a flow ϕ and express to what extend an other trajectory $\phi(t, t_0, x_0)$, whose initial state x_0 lies in the neighborhood of x^* at time t_0, remains or gets close to $\phi(t, t_0, x^*)$ for all time $t \geq t_0$.

Definition 3.1 (Lyapunov stability) Let $\phi : T \times T \times X$ be a flow and suppose that $T = \mathbb{R}$ and X is a normed vector space. The fixed point x^* is said to be

(a) *stable* (in the sense of Lyapunov) if given any $\varepsilon > 0$ and $t_0 \in T$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ (not depending on t) such that

$$\|x_0 - x^*\| \leq \delta \implies \|\phi(t, t_0, x_0) - x^*\| \leq \varepsilon \text{ for all } t \geq t_0. \tag{3.1.2}$$

(b) *attractive* if for all $t_0 \in T$ there exists $\delta = \delta(t_0) > 0$ with the property that

$$\|x_0 - x^*\| \leq \delta \implies \lim_{t \to \infty} \|\phi(t, t_0, x_0) - x^*\| = 0. \tag{3.1.3}$$

(c) *exponentially stable* if for all $t_0 \in T$ there exists $\delta = \delta(t_0)$, $\alpha = \alpha(t_0) > 0$ and $\beta = \beta(t_0) > 0$ such that

$$\|x_0 - x^*\| \leq \delta \implies \|\phi(t, t_0, x_0) - x^*\| \leq \beta \|x_0 - x^*\| e^{-\alpha(t - t_0)} \text{ for all } t \geq t_0. \tag{3.1.4}$$

(d) *asymptotically stable* (in the sense of Lyapunov) if it is both stable (in the sense of Lyapunov) and attractive.

(e) *unstable* if it is not stable (in the sense of Lyapunov).

(f) *uniformly stable* (in the sense of Lyapunov) if given any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ (not depending on t_0) such that (3.1.2) holds for all $t_0 \in T$.

(g) *uniformly attractive* if there exists $\delta > 0$ (not depending on t_0) such that (3.1.3) holds for all $t_0 \in T$.

(h) *uniformly exponentially stable* if there exists $\delta > 0$ (not depending on t_0) such that (3.1.3) holds for all $t_0 \in T$.

(i) *uniformly asymptotically stable* (in the sense of Lyapunov) if it is both uniformly stable (in the sense of Lyapunov) and uniformly attractive.

In words, a fixed point is stable if the graphs of all flows that initiate sufficiently close to x^* at time t_0, remain as close as desired to x^* for all time $t \geq t_0$. Stated otherwise, a fixed point is stable if the mapping $\phi(t, t_0, \cdot)$ is continuous at x^*, uniformly in $t \geq t_0$. The *region of attraction* associated with a fixed point x^* is defined to be the set of all initial states $x_0 \in X$ for which $\phi(t, t_0, x_0) \to x^*$ as $t \to \infty$. If this region does not depend on t_0, it is said to be *uniform*, if it coincides with X then x^* is *globally attractive*. Similarly, we can define the region of stability, the region of asymptotic stability and the region of exponential stability associated with x^*. Again, these regions are said to be *uniform* if they do not depend on t_0. If these regions cover the entire state space X, then the fixed point is called *globally stable*, *globally asymptotically stable*, or *globally exponentially stable*, respectively.

There exist examples of stable fixed points that are not attractive. Likewise, there exist examples of attractive fixed points that are not stable. The notion of exponential stability is the strongest in the sense that an exponentially stable fixed point is also asymptotically stable (i.e., a stable attractor). Similarly, it is easily seen that uniform exponential stability implies uniform asymptotic stability.

A set $\mathscr{S} \subset X$ is called *positive invariant* for a flow ϕ if $x_0 \in \mathscr{S}$ implies that there exists a $t_0 \in T$ such that $\phi(t, t_0, x_0) \in \mathscr{S}$ for all $t \geq t_0$. It is called a *negative invariant set* if this condition holds for $t \leq t_0$ and it is said to be an *invariant set* if it is both positive and negative invariant. The idea of (positive, negative) invariance simply expresses the idea that a flow remains in the set once it started there at time t_0. Naturally, $\mathscr{S} \subseteq X$ is said to be an invariant set of the differential equation (3.1.1), if

it is an invariant set of its associated flow. Also, any point $x_0 \in X$ naturally generates the invariant set $\mathscr{S} = \{\phi(t,t_0,x_0) \mid t \geq t_0\}$ consisting of all points through which the flow $\phi(t,t_0,x_0)$ passes when time evolves. In particular, every fixed point x^* of (3.1.1) defines the singleton $\mathscr{S} = \{x^*\}$ as invariant set.

The following proposition gives a first good reason to avoid distinguishing all of the above notions of stability.

Proposition 3.2 *Let $\phi : T \times T \times X$ be a flow with $T = \mathbb{R}$ and suppose that x^* is a fixed point. If ϕ is linear then*

(a) *x^* is attractive if and only if x^* is globally attractive.*

(b) *x^* is asymptotically stable if and only if x^* is globally asymptotically stable.*

(c) *x^* is exponentially stable if and only if x^* is globally exponentially stable.*

If ϕ is time-invariant then

(a) *x^* is stable if and only if x^* is uniformly stable.*

(b) *x^* is asymptotically stable if and only if x^* is uniformly asymptotically stable.*

(c) *x^* is exponentially stable if and only if x^* is uniformly exponentially stable.*

Proof.   All *if* parts are trivial. To prove the *only if* parts, let ϕ be linear, and suppose that x^* is attractive. Without loss of generality we will assume that $x^* = 0$. Take $x_0 \in X$ and $\delta > 0$ as in Definition 3.1. Then there exists $\alpha > 0$ such that $\|\alpha x_0\| < \delta$ and by linearity of the flow, $\lim_{t \to \infty} \|\phi(t,t_0,x_0)\| = \alpha^{-1} \lim_{t \to \infty} \|\phi(t,t_0,\alpha x_0)\| = 0$, i.e., 0 is a global attractor. The second and third claim for linear flows is now obvious. Next, let ϕ be time-invariant and $x^* = 0$ stable. Then for $\varepsilon > 0$ and $t_0 \in T$ there exists $\delta > 0$ such that $\|x_0\| \leq \delta$ implies $\|\phi(t+\tau,t_0+\tau,x_0)\| = \|\phi(t,t_0,x_0)\| \leq \varepsilon$ for all $t \geq t_0$ and $\tau \in T$. Set $t' = t + \tau$ and $t_0' = t_0 + \tau$ to infer that $\|x_0\| \leq \delta$ implies $\|\phi(t',t_0',x_0)\| \leq \varepsilon$ for all $t' \geq t_0'$. But these are the trajectories passing through x_0 at time t_0' with t_0' arbitrary. Hence 0 is uniformly stable. The last claims follow with a similar reasoning. ∎

Definition 3.3 (Definite functions) Let $\mathscr{S} \subseteq \mathbb{R}^n$ have the point x^* in its interior and let $T \subseteq \mathbb{R}$. A function $V : \mathscr{S} \times T \to \mathbb{R}$ is said to be

(a) *positive definite (with respect to x^*)* if there exists a continuous, strictly increasing function $a : \mathbb{R}_+ \to \mathbb{R}_+$ with $a(0) = 0$ such that $V(x,t) \geq a(\|x - x^*\|)$ for all $(x,t) \in \mathscr{S} \times T$.

(b) *positive semi-definite* if $V(x,t) \geq 0$ for all $(x,t) \in \mathscr{S} \times T$.

(c) *decrescent (with respect to x^*)* if there exists a continuous, strictly increasing function $b : \mathbb{R}_+ \to \mathbb{R}_+$ with $b(0) = 0$ such that $V(x,t) \leq b(\|x - x^*\|)$ for all $(x,t) \in \mathscr{S} \times T$.

(d) *negative definite (around x^*)* or *negative semi-definite* if $-V$ is positive definite or positive semi-definite, respectively.

The terminology in Definition 3.3 is consistent with the notions of definite *matrices* that we introduced in Chapter 1. Indeed, if K is a real symmetric matrix then for all vectors x we have that

$$\lambda_{\min}(K)\|x\|^2 \ \leq \ x^\top K x \ \leq \lambda_{\max}(K)\|x\|^2.$$

Hence, the function $V(x) := x^\top K x$ is positive definite with respect to the origin if, and only if, the smallest eigenvalue $\lambda_{\min}(K)$ is positive. In turn, this is equivalent to saying that K is positive definite as a matrix (denoted $K \succ 0$). Similarly, V is negative definite with respect to the origin if and only if $\lambda_{\max}(K) < 0$ which is equivalent to $K \prec 0$.

Consider the system (3.1.1) and suppose that x^* is an equilibrium point. Let $\mathscr{S}$ be a set which has x^* in its interior and suppose that $V : \mathscr{S} \times T \to \mathbb{R}$ has continuous partial derivatives (i.e., V is continuously differentiable). Consider, for $(x_0,t_0) \in \mathscr{S} \times T$, the function $\overline{V} : T \to X$ defined by the composition

$$\overline{V}(t) := V(\phi(t,t_0,x_0),t).$$

This function is differentiable and its derivative reads

$$\frac{d\overline{V}}{dt}(t) = \partial_x V(\phi(t,t_0,x_0),t) f(\phi(t,t_0,x_0),t) + \partial_t V(\phi(t,t_0,x_0),t).$$

Now introduce the mapping $V' : \mathscr{S} \times T \to \mathbb{R}$ by setting

$$V'(x,t) := \partial_x V(x,t) f(x,t) + \partial_t V(x,t). \tag{3.1.5}$$

V' is called the *derivative of V along trajectories of* (3.1.1) and, by construction, $\dot{\overline{V}}(t) = V'(\phi(t,t_0,x_0),t)$ for all $t \in T$. It is very important to observe that V' not only depends on V but also on the differential equation (3.1.1). It is rather common to write $\dot{V}$ for V' and even more common to confuse $\dot{\overline{V}}$ with V'. Formally, these objects are truly different as $\overline{V}$ is a function of time, whereas V' is a function of state and time.

The main stability results for autonomous systems of the form (3.1.1) are summarized in the following result.

Theorem 3.4 (Lyapunov theorem) *Consider the differential equation* (3.1.1) *and let $x^* \in X$ be an equilibrium point which belongs to the interior of a set $\mathscr{S}$.*

(a) *If there exists a positive definite, continuously differentiable function $V : \mathscr{S} \times T \to \mathbb{R}$ with $V(x^*,t) = 0$ and V' negative semi-definite, then x^* is stable. If, in addition, V is decrescent, then x^* is uniformly stable.*

(b) *If there exists a positive definite decrescent and continuously differentiable function $V : \mathscr{S} \times T \to \mathbb{R}$ with $V(x^*,t) = 0$ and V' negative definite, then x^* is uniformly asymptotically stable.*

Proof. 1. Let $t_0 \in T$ and $\varepsilon > 0$. Since $V(\cdot, t_0)$ is continuous at x^* and $V(x^*, t_0) = 0$, there exists $\delta > 0$ such that $V(x_0, t_0) \leq a(\varepsilon)$ for every $x_0 \in \mathscr{S}$ with $\|x_0 - x^*\| < \delta$. Since V is a positive definite and $dV(x(t), t)/dt \leq 0$, we have that for every $x_0 \in \mathscr{S}$ with $\|x_0 - x^*\| < \delta$ and $t \geq t_0$:

$$a(\|x(t) - x^*\|) \leq V(x(t), t) \leq V(x_0, t_0) \leq a(\varepsilon).$$

where we denoted $x(t) = \phi(t, t_0, x_0)$. Since a is strictly increasing, this implies (3.1.2), i.e., x^* is stable. If, in addition, V is decrescent, then $V(x, t) \leq b(\|x - x^*\|)$ for all $(x, t) \in \mathscr{S} \times T$. Apply the previous argument with δ such that $b(\delta) \leq a(\varepsilon)$. Then δ is independent of t_0 and $V(x_0, t_0) \leq b(\delta) \leq a(\varepsilon)$ for every $(x_0, t_0) \in \mathscr{S} \times T$ such that $\|x_0 - x^*\| < \delta$. Hence, (3.1.2) holds for all t_0.

2. By item 1, x^* is uniformly stable. It thus suffices to show that x^* is uniformly attractive. Let $\delta > 0$ be such that all x_0 with $\|x_0 - x^*\| < \delta$ belong to $\mathscr{S}$. Since x^* is an interior point of $\mathscr{S}$ such δ obviously exists. Let x_0 satisfy $\|x_0 - x^*\| < \delta$ and let $t_0 \in T$. Under the given hypothesis, there exist continuous, strictly increasing functions a, b and c such that $a(\|x - x^*\|) \leq V(x, t) \leq b(\|x - x^*\|)$ and $\dot{V}(x, t) \geq -c(\|x - x^*\|)$ for all $(x, t) \in \mathscr{S} \times T$. Let $\varepsilon > 0$, $\gamma > 0$ such that $b(\gamma) < a(\varepsilon)$, and $t_1 > t_0 + b(\delta)/c(\gamma)$. We claim that there exists $\tau \in [t_0, t_1]$ such that $x(\tau) := \phi(\tau, t_0, x_0)$ satisfies $\|x(\tau) - x^*\| \leq \gamma$. Indeed, if no such τ exists, integration of both sides of the inequality $\dot{V}(x(t), t) \leq -c(\|x(t) - x^*\|)$ yields that

$$V(x(t_1), t_1) \leq V(x_0, t_0) - \int_{t_0}^{t_1} c(\|x(t) - x^*\|) \mathrm{d}t$$

$$< b(\|x_0 - x^*\|) - (t_1 - t_0)c(\gamma) \ < \ b(\delta) - \frac{b(\delta)}{c(\gamma)}c(\gamma) = 0,$$

which contradicts the assumption that $V(x(t_1), t_1) \geq 0$. Consequently, it follows from the hypothesis that for all $t \geq \tau$:

$$a(\|x(t) - x^*\|) \leq V(x(t), t) \leq V(x(\tau), \tau) \leq b(\|x(\tau) - x^*\|) \leq b(\gamma) \leq a(\varepsilon).$$

Since a is strictly increasing, this yields that $\|x(t) - x^*\| \leq \varepsilon$ for all $t \geq \tau$. As ε is arbitrary, this proves (3.1.3) for all t_0. ∎

Functions V that satisfy either of the properties of Theorem 3.4 are generally referred to as *Lyapunov functions*. The main implication of Theorem 3.4 is that stability of equilibrium points of differential equations of the form (3.1.1) can be verified by searching for suitable Lyapunov functions.

3.1.2 Stability of interconnected systems

The class of autonomous systems that is studied in this section may seem rather restricted at first sight. However, also for applications in control, the importance of autonomous systems becomes apparent when considering interconnections of non-autonomous dynamical systems. A full interconnection of two time-invariant non-autonomous and possibly nonlinear systems S_1 and S_2 of the form (2.4.1) is autonomous (when well-posed). In addition, the following result is immediate from Corollary 2.16 of Chapter 2.

Theorem 3.5 *Let* S_1 *and* S_2 *be dynamical systems of the form* (2.4.1) *with* $\dim(W_i) = \dim(Z_i) = 0$, $i = 1, 2$. *Suppose that both* (S_1, s) *and* $(S_2, -s)$ *are dissipative for some supply function* s *and suppose that each system* S_i, $i = 1, 2$, *admits a storage function* $V_i : X_i \to \mathbb{R}$ *that is continuous and positive definite with respect to an equilibrium point* x_i^* *of* S_i. *Then* (x_1^*, x_2^*) *is a stable equilibrium point of the full interconnection* $S_1 \sqcap S_2$. *Moreover, the function* $V : X_1 \times X_2 \to \mathbb{R}$ *defined by*

$$V(x_1, x_2) := V_1(x_1) + V_2(x_2)$$

is a Lyapunov function for the equilibrium point in the sense that V *is positive definite and* V' *negative semi-definite.*

Proof. Obviously, V is positive definite at $x^* = (x_1^*, x_2^*)$. By Corollary 2.16 of Chapter 2, V' is negative semi-definite at x^*. The stability of the equilibrium point x^* follows from Theorem 3.4. $\blacksquare$

Hence, storage functions, introduced in Chapter 2, and Lyapunov functions are closely related. Also compare (3.1.5) with the differential dissipation inequality (2.2.3) from Chapter 2. Understanding this relation is worthwhile, as it often helps to construct Lyapunov functions on the basis of physical properties of the system, or to decide about the stability of a system, simply by inspecting the interconnection structure of components. For us, Theorem 3.5 will be at the basis of many robust stabilization results that are yet to come. See Chapter 7.

As an example, a damped pendulum described by the nonlinear differential equations

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin(x_1) - bx_2$$

has all points $(k\pi, 0)$ with $k \in \mathbb{Z}$, corresponding to the vertical positions of the pendulum, as its fixed points. The function $V(x_1, x_2) := \frac{1}{2}x_2^2 + (1 - \cos(x_1))$, representing the sum of kinetic and potential mechanical energy in the system, vanishes at the fixed points $(k\pi, 0)$ with k even and is a Lyapunov function with $V'(x_1, x_2) := \tilde{V}(x_1, x_2) = -bx_2^2$ negative definite provided that the damping coefficient $b > 0$. Hence, the fixed points $(k\pi, 0)$ with k even are stable equilibria for any $b > 0$. In fact, these points are uniformly asymptotically stable. Moreover, V satisfies the dissipation inequality (2.2.2) with supply function $s = 0$.

3.1.3 Nominal stability of linear systems

Let us consider the *linear autonomous system*

$$\dot{x} = Ax \tag{3.1.6}$$

where $A \in \mathbb{R}^{n \times n}$. Then (3.1.6) defines a linear flow. Suppose that (3.1.6) is obtained as the *linearization* of $f : X \to X$ around an equilibrium point $x^* \in X$ of (3.1.1). Precisely, the first order approximation of f at the equilibrium point $x^* \in X$ is given by

$$f(x) \approx f(x^*) + \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x^*) & \cdots & \frac{\partial f_1}{\partial x_n}(x^*) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(x^*) & \cdots & \frac{\partial f_n}{\partial x_n}(x^*) \end{pmatrix} (x - x^*)$$

where we assume that f is at least once differentiable and where f_k denotes the kth component of f. Assuming (without loss of generality) that $x^* = 0$, the linearization of f around $x^* = 0$ is defined by the system (3.1.6) with

$$A := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x^*) & \cdots & \frac{\partial f_1}{\partial x_n}(x^*) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(x^*) & \cdots & \frac{\partial f_n}{\partial x_n}(x^*) \end{pmatrix}.$$

In more compact notation, $A = \partial_x f(x^*)$, which is usually referred to as the *Jacobian* of f at x^*. As we will see, the stability of the equilibrium point x^* of the nonlinear flow (3.1.1) can be inferred from the stability of the origin in the linearized system (3.1.6). Define the quadratic form $V : X \to \mathbb{R}$ by setting

$$V(x) = x^\top K x$$

with $K \succ 0$. This function serves as a quadratic Lyapunov function for the linear flow (3.1.6). Indeed, V is continuous at $x^* = 0$, it assumes a strong local minimum at $x = 0$ (actually this is a strong global minimum of V), while the derivative of $V(\cdot)$ in the direction of the vector field Ax is given by

$$V'(x) = \partial_x V(x) Ax = 2x^\top K Ax = x^\top [A^\top K + KA] x.$$

If $A^\top K + KA \prec 0$ we infer that V' is negative definite which, using Theorem 3.4, guarantees that the origin is an asymptotically stable equilibrium point of (3.1.6). In fact, many more conclusions can be drawn:

Proposition 3.6 *Let the system* (3.1.6) *be a linearization of* (3.1.1) *at the equilibrium x^* of* (3.1.1). *The following statements are equivalent.*

(a) The origin is an asymptotically stable equilibrium for (3.1.6).

(b) The origin is a globally asymptotically stable equilibrium for (3.1.6).

(c) All eigenvalues $\lambda(A)$ of A have strictly negative real part (i.e., A is Hurwitz).

(d) The linear matrix inequalities

$$A^\top K + KA \prec 0, \qquad K \succ 0$$

are feasible.

Moreover, if one of these statements hold, then the equilibrium x^* of the flow (3.1.1) *is asymptotically stable.*

As the most important implication of Proposition 3.6, asymptotic stability of the equilibrium x^* of the nonlinear system (3.1.1) can be concluded from the asymptotic stability of its linearization at x^*.

3.2 Generalized stability regions for LTI systems

As we have seen, the autonomous linear system

$$\dot{x} = Ax$$

is asymptotically stable if and only if all eigenvalues of A lie in $\mathbb{C}^-$, the open left half complex plane. For many applications in control and engineering we may be interested in more general stability regions. Let us define a *stability region* as a subset $\mathbb{C}_{\text{stab}} \subseteq \mathbb{C}$ with the following two properties

$$\begin{cases} \text{Property 1:} & \lambda \in \mathbb{C}_{\text{stab}} \implies \bar{\lambda} \in \mathbb{C}_{\text{stab}} \\ \text{Property 2:} & \mathbb{C}_{\text{stab}} \text{ is convex .} \end{cases}$$

Typical examples of common stability sets include

$\mathbb{C}_{\text{stab } 1} = \mathbb{C}^-$	open left half complex plane
$\mathbb{C}_{\text{stab } 2} = \mathbb{C}$	no stability requirement
$\mathbb{C}_{\text{stab } 3} = \{s \in \mathbb{C} \mid \text{Re}(s) < -\alpha\}$	guaranteed damping
$\mathbb{C}_{\text{stab } 4} = \{s \in \mathbb{C} \mid |s| < r\}$	circle centered at origin
$\mathbb{C}_{\text{stab } 5} = \{s \in \mathbb{C} \mid \alpha_1 < \text{Re}(s) < \alpha_2\}$	vertical strip
$\mathbb{C}_{\text{stab } 6} = \{s \in \mathbb{C} \mid \text{Re}(s)\tan(\theta) < -|\text{Im}(s)|\}$	conic stability region.

Here, $\theta \in (0, \pi/2)$ and $r, \alpha, \alpha_1, \alpha_2$ are real numbers. We consider the question whether we can derive a feasibility test to verify whether the eigen-modes of the system $\dot{x} = Ax$ belong to either of these sets. This can indeed be done in the case of the given examples. To see this, let us introduce the notion of an LMI-region as follows:

Definition 3.7 For a real symmetric matrix $P \in \mathbb{S}^{2m \times 2m}$, the set of complex numbers

$$L_P := \left\{ s \in \mathbb{C} \mid \begin{pmatrix} I \\ sI \end{pmatrix}^* P \begin{pmatrix} I \\ sI \end{pmatrix} \prec 0 \right\}$$

is called an *LMI region*.

If P is partitioned according to $P = \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix}$, then an LMI region is defined by those points $s \in \mathbb{C}$ for which

$$Q + sS + \bar{s}S^\top + \bar{s}Rs \prec 0.$$

All of the above examples fit in this definition. Indeed, by setting

$$P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_3 = \begin{pmatrix} 2\alpha & 1 \\ 1 & 0 \end{pmatrix}$$

$$P_4 = \begin{pmatrix} -r^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 2\alpha_1 & 0 & -1 & 0 \\ 0 & -2\alpha_2 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad P_6 = \begin{pmatrix} 0 & 0 & \sin(\theta) & \cos(\theta) \\ 0 & 0 & -\cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) & 0 & 0 \\ \cos(\theta) & \sin(\theta) & 0 & 0 \end{pmatrix}$$

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we obtain that $\mathbb{C}_{\text{stab}\, i} = L_{P_i}$. More specifically, LMI regions include regions bounded by circles, ellipses, strips, parabolas and hyperbolas. Since any finite intersection of LMI regions is again an LMI region one can virtually approximate any convex region in the complex plane.

To present the main result of this section, we will need to introduce the notation for *Kronecker products*. Given two matrices $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{k \times \ell}$, the Kronecker product of A and B is the $mk \times n\ell$ matrix

$$A \otimes B = \begin{pmatrix} A_{11}B & \dots & A_{1n}B \\ \vdots & & \vdots \\ A_{m1}B & \dots & A_{mn}B \end{pmatrix}$$

Some properties pertaining to the Kronecker product are as follows

- $1 \otimes A = A = A \otimes 1$.

- $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$.

- $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

- $(A \otimes B)^* = A^* \otimes B^*$.

- $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

- in general $A \otimes B \neq B \otimes A$.

These properties are easily verified and we will not prove them here. Stability regions described by LMI regions lead to the following interesting generalization of the Lyapunov inequality.

Theorem 3.8 *All eigenvalues of $A \in \mathbb{R}^{n \times n}$ are contained in the LMI region*

$$\left\{ s \in \mathbb{C} \mid \begin{pmatrix} I \\ sI \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} I \\ sI \end{pmatrix} \prec 0 \right\}$$

if and only if there exists $K \succ 0$ such that

$$\begin{pmatrix} I \\ A \otimes I \end{pmatrix}^* \begin{pmatrix} K \otimes Q & K \otimes S \\ K \otimes S^\top & K \otimes R \end{pmatrix} \begin{pmatrix} I \\ A \otimes I \end{pmatrix} \prec 0$$

Note that the latter is an LMI in K and that the Lyapunov theorem (Theorem 3.4) corresponds to taking $Q = 0$, $S = I$ and $R = 0$. Among the many interesting special cases of LMI regions that are covered by Theorem 3.8, we mention the stability set $\mathbb{C}_{\text{stab}\, 4}$ with $r = 1$ used for the characterization of stability of the *discrete time* system $x(t+1) = Ax(t)$. This system is stable if and only if the eigenvalues of A are inside the unit circle. Equivalently, $\lambda(A) \in L_P$ with $P = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, which by Theorem 3.8 is equivalent to saying that there exist $K \succ 0$ such that

$$\begin{pmatrix} I \\ A \end{pmatrix}^* \begin{pmatrix} -K & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} I \\ A \end{pmatrix} = A^* K A - K \prec 0.$$

3.3 Nominal performance and LMI's

In this section we will use the results on dissipative systems of Chapter 2 to characterize a number of relevant performance criteria for dynamical systems. In view of forthcoming chapters we consider the system

$$
\mathsf{S}: \quad \begin{cases} \dot{x} = Ax + Bw \qquad x(0) = x_0 \\ z = Cx + Dw \end{cases} \tag{3.3.1}
$$

where $x(t) \in X = \mathbb{R}^n$ is the state, $w(t) \in W = \mathbb{R}^m$ the input and $z(t) \in Z = \mathbb{R}^p$ the output. Here, $t \in \mathbb{R}_+$. Let $T(s) = C(Is - A)^{-1}B + D$ denote the corresponding transfer function and assume throughout this section that the system is asymptotically stable (i.e., the eigenvalues of A are in the open left-half complex plane). We will view w as an input variable (a 'disturbance') whose effect on the output z (an 'error indicator') we wish to minimize. There are various ways to quantify the effect of w on z. For example, for a given input w, and for suitable signal norms, the quotient $\|z\|/\|w\|$ indicates the relative gain which the input w has on the output z. More generally, the *worst case gain* of the system is the quantity

$$
\|T\| := \sup_{0 < \|w\| < \infty} \frac{\|z\|}{\|w\|} \tag{3.3.2}
$$

which, of course, depends on the chosen signal norms. Other indicators for nominal performance could be the energy in the impulse response of the system, the (asymptotic) variance of the output when the system is fed with inputs with a prescribed stochastic nature, percentage overshoot in step responses, etc.

3.3.1 Quadratic nominal performance

We start this section by reconsidering Theorem 2.9 from Chapter 2. The following proposition is obtained by rephrasing Theorem 2.9 with a sign-changed quadratic storage function.

Proposition 3.9 *Consider the system* (3.3.1) *with transfer function T and let*

$$
P := \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix}
$$

be a real symmetric matrix in $\mathbb{S}^{m+n}$. Suppose that A is Hurwitz. Then the following statements are equivalent.

(a) *there exists $\varepsilon > 0$ such that for $x(0) = 0$ and for all $w \in \mathscr{L}_2$*

$$
\int_0^\infty \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} dt \; \leq \; -\varepsilon^2 \int_0^\infty w(t)^\top w(t) dt \tag{3.3.3}
$$

(b) for all $\omega \in \mathbb{R} \cup \{\pm\infty\}$ there holds

$$\begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} \prec 0.$$

(c) there exists $K = K^\top$ such that

$$F(K) = \begin{pmatrix} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{pmatrix}^\top \left(\begin{array}{cc|cc} 0 & K & 0 & 0 \\ K & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S \\ 0 & 0 & S^\top & R \end{array} \right) \begin{pmatrix} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{pmatrix} \quad \prec \quad 0.$$

Proof. We run the circle $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$. Suppose that item 1 holds. Since the system is stable, any $w \in \mathcal{L}_2$ defines a unique state trajectory $x(t) = \int_0^t \exp(A(t-\tau))Bw(\tau)\mathrm{d}\tau$, $t \in \mathbb{R}_+$, that satisfies (3.3.1) with $x(0) = 0$, and belongs to $\mathcal{L}_2$. Consequently, also $z \in \mathcal{L}_2$ which shows that the indefinite integrals in (3.3.3) are well defined and can be rewritten as

$$\int_0^\infty \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^\top \begin{pmatrix} Q + \varepsilon^2 I & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} \mathrm{d}t \leq 0 \qquad \text{for all } w \in \mathcal{L}_2.$$

Using Parseval, this is equivalent to saying that

$$\int_{-\infty}^\infty \hat{w}(i\omega)^* \begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \begin{pmatrix} Q + \varepsilon^2 I & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} \hat{w}(i\omega) \mathrm{d}\omega \leq 0, \qquad \text{for all } \hat{w} \in \mathcal{L}_2.$$

Since $\hat{w}$ is arbitrary, this yields

$$\begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \begin{pmatrix} Q + \varepsilon^2 I & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} \preccurlyeq 0 \qquad \text{for all } \omega \in \mathbb{R} \cup \{\pm\infty\}$$

which, in turn, implies the strict inequality of item (2).

The implication $(2) \Rightarrow (3)$ is a consequence of the Kalman-Yakubovich-Popov Lemma 2.11 and has been established in Theorem 2.9.

Remains to show that (3) implies (1). By Theorem 2.9, $F(K) \prec 0$ implies that the strict dissipation inequality (2.2.4) holds with $V(x) := x^\top K x$ and some $\varepsilon > 0$. In (2.2.4), let $t_0 = 0$, $x(0) = 0$, $t_1 \to \infty$ and observe that the indefinite integral remains well defined as A is Hurwitz. This yields item (1) as desired. ∎

This result characterizes *quadratic performance* of stable systems (3.3.1) in the sense that it provides necessary and sufficient conditions for the quadratic performance function $J := \int_0^\infty \mathrm{s}(w(t), z(t))\mathrm{d}t$ to be strictly negative for all square integrable trajectories of a stable system. Proposition (3.9) provides a frequency domain inequality and a linear matrix inequality to characterize quadratic performance. This very general result proves useful in quite a number of important special cases, which we describe below.

Remark 3.10 Proposition 3.9 does not characterize quadratic performance *and* stability of the system. By Proposition 3.6, A is Hurwitz if and only if there exists $K \succ 0$ with $A^\top K + KA \prec 0$. In many practical cases the performance criterion in item 3 implicitly involves the inequality $A^\top K + KA \prec 0$, for example if $R \succcurlyeq 0$. Indeed, if $R \succcurlyeq 0$ then $F(K) \prec 0$ implies $A^\top K + KA \prec 0$ so that A is Hurwitz if and only if $K \succ 0$ in the LMI of item (3). Therefore, as long as $C^\top RC \succcurlyeq 0$ we can characterize quadratic performance *and* stability with the same LMI by just including the extra condition $K \succ 0$ in item (3).

3.3.2 H_∞ nominal performance

A popular performance measure of a stable linear time-invariant system is the H_∞ norm of its transfer function. It is defined as follows. Consider the system (3.3.1) together with its transfer function T. Assume the system to be asymptotically stable. In that case, $T(s)$ is bounded for all $s \in \mathbb{C}$ with positive real part. By this, we mean that the largest singular value $\sigma_{\max}(T(s))$ is finite for all $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$. This is an example of an H_∞ function. To be slightly more formal on this class of functions, let $\mathbb{C}^+$ denote the set of complex numbers with positive real part. The *Hardy space H_∞* consists of all complex valued functions $T : \mathbb{C}^+ \to \mathbb{C}^{p \times m}$ which are analytic and for which

$$\|T\|_\infty := \sup_{s \in \mathbb{C}^+} \sigma_{\max}(T(s)) < \infty.$$

The left-hand side of this expression satisfies the axioms of a norm and defines the H_∞ norm of T. Although H_∞ functions are defined on the right-half complex plane, it can be shown that each such function has a unique extension to the imaginary axis (which is usually also denoted by T) and that the H_∞ norm of T satisfies

$$\|T\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(T(i\omega)).$$

In words, the H_∞ norm of a transfer function is the supremum of the maximum singular value of the frequency response of the system.

Remark 3.11 Various graphical representations of frequency responses are illustrative to investigate system properties like bandwidth, gains, etc. Probably the most important one is a plot of the singular values $\sigma_j(T(i\omega))$ $(j = 1, \ldots, \min(m, p))$ viewed as function of the frequency $\omega \in \mathbb{R}$. For single-input single-output systems there is only one singular value and $\sigma(T(i\omega)) = |T(i\omega)|$. A *Bode diagram* of the system is a plot of the mapping $\omega \mapsto |T(i\omega)|$ and provides useful information to what extent the system amplifies purely harmonic input signals with frequencies $\omega \in \mathbb{R}$. In order to interpret these diagrams one usually takes logarithmic scales on the ω axis and plots $20\log_{10}(T(j\omega))$ to get units in decibels dB. The H_∞ norm of a transfer function is then nothing else than the highest peak value which occurs in the Bode plot. In other words it is the largest gain if the system is fed with harmonic input signals.

The H_∞ norm of a stable linear system admits an interpretation in terms of dissipativity of the system with respect to a specific quadratic supply function. Alternatively, the H_∞ norm of a transfer function allows an important interpretation as the induced norm (3.3.2) with respect to square integrable signal norms. The relevant properties are summarized in the following result.

Proposition 3.12 *Suppose that A in (3.3.1) is Hurwitz and let $\gamma > 0$. Then the following statements are equivalent.*

(a) $\|T\|_\infty < \gamma$.

(b) *the induced system norm*

$$\|T\|_{2,2} := \sup_{0 < \|w\|_2 < \infty} \frac{\|z\|_2}{\|w\|_2} < \gamma$$

where z is the output of (3.3.1) subject to input w and initial condition $x(0) = 0$.

(c) *The system (3.3.1) is strictly dissipative with respect to the supply function* $s(w,z) = \gamma \|w\|^2 - \frac{1}{\gamma}\|z\|^2$.

(d) *there exists a solution $K = K^\top$ to the LMI*

$$\begin{pmatrix} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{pmatrix}^\top \left(\begin{array}{cc|cc} 0 & K & 0 & 0 \\ K & 0 & 0 & 0 \\ \hline 0 & 0 & -\gamma I & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma}I \end{array} \right) \begin{pmatrix} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{pmatrix} \prec 0 \qquad (3.3.4)$$

(e) *For all $\omega \in \mathbb{R} \cup \{\infty\}$ one has $T(i\omega)^* T(i\omega) \prec \gamma^2 I$.*

Proof. Apply Proposition 3.9 with

$$\begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} = \begin{pmatrix} -\gamma I & 0 \\ 0 & \frac{1}{\gamma}I \end{pmatrix}.$$

The last item is obtained by recalling that $\sigma_{\max}^2(T(i\omega)) = \lambda_{\max}(T(\omega)^* T(i\omega))$. ∎

For a stable system, the H_∞ norm of the transfer function therefore coincides with the $\mathscr{L}_2$-induced norm of the input-output operator associated with the system. In view of (3.3.2) we therefore have that $\|T\|_\infty = \|T\|_{2,2}$, i.e., the H_∞ norm of a transfer function equals the $\mathscr{L}_2$ induced norm of the input-output mapping that the system defines. Using the Kalman-Yakubovich-Popov lemma, this yields a practical LMI feasibility test to verify whether or not the H_∞ norm of the transfer function T is bounded by γ. The LMI (3.3.4) can be written in various equivalent forms. In view of forthcoming chapters, we list a few equivalent feasibility tests here. That is, (3.3.4) is feasible as an LMI in the unknown K if and only if either of the following inequalities is feasible

$$\begin{pmatrix} A^\top K + KA + C^\top C & KB + C^\top D \\ B^\top K + D^\top C & D^\top D - \gamma^2 I \end{pmatrix} \prec 0$$

$$A^\top K + KA + C^\top C + (KB + C^\top D)(\gamma^2 I - D^\top D)^{-1}(B^\top K + D^\top C) \prec 0$$

$$\begin{pmatrix} A^\top K + KA & KB & C^\top \\ B^\top K & -\gamma I & D^\top \\ C & D & -\gamma I \end{pmatrix} \prec 0.$$

3.3.3 H_2 nominal performance

The Hardy space H_2 consists of all complex valued functions which are analytic in $\mathbb{C}^+$ and for which

$$\|T\|_{H_2} := \sqrt{\frac{1}{2\pi} \sup_{\sigma > 0} \operatorname{trace} \int_{-\infty}^{\infty} T(\sigma + i\omega)[T(\sigma + i\omega)]^* \, d\omega}$$

is finite. This defines the H_2 norm of T. This 'cold-blooded' definition may seem little appealing at first sight but, in fact, it has nice and important system theoretic interpretations. As in H_∞, it can be shown that each function in H_2 has a unique extension to the imaginary axis, which we also denote by T, and that, in fact, the H_2 norm satisfies

$$\|T\|_{H_2}^2 = \frac{1}{2\pi} \operatorname{trace} \int_{-\infty}^{\infty} T(i\omega)T(i\omega)^* \, d\omega. \tag{3.3.5}$$

We will first give an interpretation of the H_2 norm of a system in terms of its *impulsive behavior*. Consider the system (3.3.1) and suppose that we are interested only in the impulse responses of this system. This means, that we take impulsive inputs[1] of the form

$$w(t) = \delta(t)e_j$$

where e_j is the jth basis vector in the standard basis of the input space $\mathbb{R}^m$, $(j = 1, \ldots, m)$. The output z^j which corresponds to the input w and initial condition $x(0) = 0$ is uniquely defined and given by

$$z^j(t) = \begin{cases} C\exp(At)Be_j & \text{for } t > 0 \\ De_j\delta(t) & \text{for } t = 0 \\ 0 & \text{for } t < 0 \end{cases}.$$

Since the system is assumed to be stable, the outputs z^j are square integrable for all $j = 1, \ldots, m$, provided that $D = 0$. Hence, $\|z^j\|_2$ is well defined provided that $D = 0$. Summing up the norms of the outputs gives

$$\sum_{j=1}^{m} \|z^j\|_2^2 = \operatorname{trace} \int_0^{\infty} B^\top \exp(A^\top t)C^\top C\exp(At)B \, dt$$

$$= \operatorname{trace} \int_0^{\infty} C\exp(At)BB^\top \exp(A^\top t)C^\top \, dt.$$

Long ago, Parseval taught us that the latter expression is equal to

$$\frac{1}{2\pi} \operatorname{trace} \int_{-\infty}^{\infty} T(i\omega)T(i\omega)^* \, d\omega$$

which is $\|T\|_{H_2}^2$. Therefore, we reach the conclusion that the squared H_2 norm of a transfer function T coincides with the total 'output energy' in the impulse responses of the system. What is more,

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[1]Formally, the impulse δ is not a function and for this reason it is neither a signal. It requires a complete introduction to distribution theory to make these statements more precise, but we will not do this at this place.

this observation provides a straightforward algorithm to determine the H_2 norm of a stable rational transfer function. Indeed, associate with the system (3.3.1) the positive semi-definite matrices

$$W := \int_0^\infty \exp(At)BB^\top \exp(A^\top t)\,\mathrm{d}t$$

$$M := \int_0^\infty \exp(A^\top t)C^\top C \exp(At)\,\mathrm{d}t.$$

Then W is usually referred to as the *controllability gramian* and M as the *observability gramian* of the system (3.3.1). The gramians satisfy the matrix equations

$$AW + WA^\top + BB^\top = 0, \qquad A^\top M + MA + C^\top C = 0$$

and are, in fact, the unique solutions to these equations whenever A has its eigenvalues in $\mathbb{C}^-$ (as is assumed here). Consequently,

$$\|T\|_{H_2}^2 = \mathrm{trace}(CWC^\top) = \mathrm{trace}(B^\top MB).$$

A second interpretation of the H_2 norm makes use of stochastic assumptions on the input. Consider the system (3.3.1) and assume that the components of the input w are independent zero-mean, white noise processes. If we take the initial condition x_0 a zero mean random variable, independent of w, and with variance $\mathscr{E}(x_0 x_0^\top) = W_0$, then the state variance matrix

$$W(t) := \mathscr{E}(x(t)x(t)^\top)$$

is the solution of the matrix differential equation

$$\dot{W} = AW + WA^\top + BB^\top, \qquad W(0) = W_0.$$

Consequently, with $D = 0$, the output variance

$$\mathscr{E}(z(t)^\top z(t)) = \mathscr{E}(x(t)^\top C^\top C x(t)) = \mathscr{E}\,\mathrm{trace}(Cx(t)x(t)^\top C^\top) =$$
$$= \mathrm{trace}\,C\mathscr{E}(x(t)x(t)^\top)C^\top = \mathrm{trace}(CW(t)C^\top).$$

Since A is Hurwitz, the limit $W := \lim_{t\to\infty} W(t)$ exists and is equal to the controllability gramian of the system (3.3.1) whenever $W_0 = 0$. Consequently, the asymptotic output variance

$$\lim_{t\to\infty} \mathscr{E}(z(t)z^\top(t)) = \mathrm{trace}(CWC^\top)$$

is the square of the H_2 norm of the system. The H_2 norm therefore has an interpretation in terms of the asymptotic output variance of the system when the system is excited by white noise input signals.

The following theorem characterizes the H_2 norm in terms of linear matrix inequalities.

Proposition 3.13 *Suppose A in (3.3.1) is Hurwitz and let $T(s) = C(Is - A)^{-1}B + D$ be the corresponding transfer function. Then*

(a) $\|T\|_2 < \infty$ *if and only if* $D = 0$.

(b) If $D = 0$ *then the following statements are equivalent*

(i) $\|T\|_2 < \gamma$

(ii) there exists $X \succ 0$ *such that*

$$AX + XA^\top + BB^\top \prec 0, \qquad \text{trace}(CXC^\top) < \gamma^2.$$

(iii) there exists $Y \succ 0$ *such that*

$$A^\top Y + YA + C^\top C \prec 0, \qquad \text{trace}(B^\top YB) < \gamma^2.$$

(iv) there exist $K = K^\top \succ 0$ *and Z such that*

$$\begin{pmatrix} A^\top K + KA & KB \\ B^\top K & -\gamma I \end{pmatrix} \prec 0; \quad \begin{pmatrix} K & C^\top \\ C & Z \end{pmatrix} \succ 0; \quad \text{trace}(Z) < \gamma \qquad (3.3.6)$$

(v) there exist $K = K^\top \succ 0$ *and Z such that*

$$\begin{pmatrix} AK + KA^\top & KC^\top \\ CK & -\gamma I \end{pmatrix} \prec 0; \quad \begin{pmatrix} K & B \\ B^\top & Z \end{pmatrix} \succ 0; \quad \text{trace}(Z) < \gamma. \qquad (3.3.7)$$

Proof. The first claim is immediate from the definition of the H_2 norm. We prove the second part by showing the equivalences (i)⇔(iii) and (iii)⇔(v), respectively. The remaining implications then follow from the observation that $\|T\|_2 = \|T^*\|_2$ where $T^*(s) = B^\top(Is - A^\top)^{-1}C^\top$.

To prove that (i)⇔(iii), first infer from the discussion preceding this theorem that $\|T\|_2 < \gamma$ is equivalent to saying that there exists M such that

$$A^\top M + MA + C^\top C = 0, \quad \text{trace}(B^\top MB) < \gamma^2. \qquad (3.3.8)$$

Here, the observability gramian $M = M^\top \succ 0$ is, in fact, uniquely determined by the Lyapunov equation. To see that (3.3.8) implies (iii), first apply Proposition 3.6 to infer from the Hurwitz property of A that there exists $P \succ 0$ with $A^\top P + PA \prec 0$. If $\text{trace}(B^\top MB) < \gamma^2$ there exists $\varepsilon > 0$ such that $Y := M + \varepsilon P$ satisfies $\text{trace}(B^\top YB) < \gamma^2$. Obviously, $Y \succ 0$ and

$$A^\top Y + YA + C^\top C = A^\top M + MA + C^\top C + \varepsilon(A^\top P + PA) \prec 0,$$

which is (iii). Conversely, if (iii) holds, there exists a matrix C_0 such that

$$A^\top Y + YA + C^\top C + C_0^\top C_0 = 0, \quad \text{trace}(B^\top YB) < \gamma^2$$

which, by (3.3.8), shows that $\|T_e\|_2 < \gamma$ where $T_e = \text{col}(T, T_0)$ with $T_0(s) = C_0(Is - A)^{-1}B$. But then $\|T\|_2 < \gamma$ which gives (i).

To prove that (iii)⇔(v), define $K = \gamma Y^{-1}$ and let $\varepsilon > 0$ be such that $Z := \frac{1}{\gamma}B^{\top}YB + \varepsilon I$ satisfies trace$(Z) < \gamma$. Since (iii) is equivalent to trace$(\frac{1}{\gamma}B^{\top}YB) < \gamma$, such ε exists. Then $K \succ 0$ and we infer that (iii) is equivalent to saying that there exists $K \succ 0$ such that

$$AK + KA^{\top} + \frac{1}{\gamma}KC^{\top}CK = \sqrt{\gamma}Y^{-1}\left[A^{\top}Y + YA + C^{\top}C\right]Y^{-1}\sqrt{\gamma} \prec 0$$

$$Z \succ \frac{1}{\gamma}B^{\top}YB = B^{\top}K^{-1}B$$

$$\text{trace}(Z) < \gamma.$$

Now, use Schur complements for the first two inequalities to see that the latter is equivalent to (3.3.7) for some $K \succ 0$ which is (v). ∎

Interpretation 3.14 The smallest possible upperbound of the H_2-norm of the transfer function can be calculated by minimizing the criterion trace(Z) over the variables $K \succ 0$ and Z that satisfy the LMI's defined by the first two inequalities in (3.3.6) or (3.3.7).

3.3.4 Generalized H_2 nominal performance

Consider again the system (3.3.1) and suppose that $x(0) = 0$ and that A is Hurwitz. Recall that $\|T\|_{H_2} < \infty$ if and only if $D = 0$. The system then defines a bounded operator from $\mathscr{L}_2$ inputs to $\mathscr{L}_{\infty}$ outputs. That is, for any input w for which $\|w\|_2^2 := \int_0^{\infty}\|w(t)\|^2\mathrm{d}t < \infty$ the corresponding output z belongs to $\mathscr{L}_{\infty}$, the space of signals $z : \mathbb{R}_+ \to \mathbb{R}^p$ of finite *amplitude*[2]

$$\|z\|_{\infty} := \sup_{t \geq 0}\sqrt{\langle z(t), z(t)\rangle}.$$

With reference to (3.3.2), the $\mathscr{L}_2$-$\mathscr{L}_{\infty}$ induced norm (or 'energy to peak' norm) of the system is defined as

$$\|T\|_{2,\infty} := \sup_{0 < \|w\|_2 < \infty}\frac{\|z\|_{\infty}}{\|w\|_2}$$

and satisfies

$$\|T\|_{2,\infty}^2 = \frac{1}{2\pi}\lambda_{\max}\left(\int_{-\infty}^{\infty}T(i\omega)T(i\omega)^*\mathrm{d}\omega\right) \tag{3.3.9}$$

where $\lambda_{\max}(\cdot)$ denotes maximum eigenvalue. Note that when z is scalar valued, the latter expression reduces to the H_2 norm, i.e, for systems with scalar valued output variables

$$\|T\|_{2,\infty} = \|T\|_{H_2},$$

which is the reason why we refer to (3.3.9) as a *generalized H_2 norm*. The following result characterizes an upperbound on this quantity.

---

[2]An alternative and more common definition for the $\mathscr{L}_{\infty}$ norm of a signal $z : \mathbb{R} \to \mathbb{R}^p$ is $\|z\|_{\infty} := \max_{j=1,\ldots,p}\sup_{t \geq 0}|z_j(t)|$. For scalar valued signals this coincides with the given definition, but for non-scalar signals this is a different signal norm. When equipped with this alternative amplitude norm of output signals, the characterization (3.3.9) still holds with $\lambda_{\max}(\cdot)$ redefined as the *maximal entry on the diagonal* of its argument. See [40] for details.

Proposition 3.15 *Suppose A in (3.3.1) is Hurwitz and that $D = 0$. Then $\|T\|_{2,\infty} < \gamma$ if and only if there exists a solution $K \succ 0$ to the LMI's*

$$\begin{pmatrix} A^\top K + KA & KB \\ B^\top K & -\gamma I \end{pmatrix} \prec 0; \qquad \begin{pmatrix} K & C^\top \\ C & \gamma I \end{pmatrix} \succ 0 \qquad (3.3.10)$$

Proof. Firstly, infer from Theorem 2.9 that the existence of $K \succ 0$ with

$$\begin{pmatrix} A^\top K + KA & KB \\ B^\top K & -I \end{pmatrix} \prec 0$$

is equivalent to the dissipativity of the system (3.3.1) with respect to the supply function $s(w, z) = w^\top w$. Equivalently, for all $w \in \mathscr{L}_2$ and $t \geq 0$ there holds

$$x(t)^\top K x(t) \leq \int_0^t w(\tau)^\top w(\tau)\, \mathrm{d}\tau.$$

Secondly, using Schur complements, the LMI

$$\begin{pmatrix} K & C^\top \\ C & \gamma^2 I \end{pmatrix} \succ 0$$

is equivalent to the existence of an $\varepsilon > 0$ such that $C^\top C \prec (\gamma^2 - \varepsilon^2)K$. Together, this yields that for all $t \geq 0$

$$\langle z(t), z(t) \rangle = x(t)^\top C^\top C x(t) \leq (\gamma^2 - \varepsilon^2) x(t)^\top K x(t)$$

$$\leq (\gamma^2 - \varepsilon^2) \int_0^t w(\tau)^\top w(\tau)\, \mathrm{d}\tau.$$

$$\leq (\gamma^2 - \varepsilon^2) \int_0^\infty w(\tau)^\top w(\tau)\, \mathrm{d}\tau.$$

Take the supremum over $t \geq 0$. This yields the existence of $\varepsilon > 0$ such that for all $w \in \mathscr{L}_2$

$$\|z\|_\infty^2 \ \leq \ (\gamma^2 - \varepsilon^2)\|w\|_2^2.$$

Dividing the latter expression by $\|w\|_2^2$ and taking the supremum over all $w \in \mathscr{L}_2$ then yields the result. ∎

3.3.5 L_1 or peak-to-peak nominal performance

Consider the system (3.3.1) and assume again that the system is stable. For fixed initial condition $x(0) = 0$ this system defines a mapping from bounded amplitude inputs $w \in \mathscr{L}_\infty$ to bounded amplitude outputs $z \in \mathscr{L}_\infty$ and a relevant performance criterion is the 'peak-to-peak' or $\mathscr{L}_\infty$-induced norm of this mapping

$$\|T\|_{\infty,\infty} := \sup_{0 < \|w\|_\infty < \infty} \frac{\|z\|_\infty}{\|w\|_\infty}.$$

The following result gives a sufficient condition for an upperbound γ of the peak-to-peak gain of the system.

Proposition 3.16 *If there exists $K \succ 0$, $\lambda > 0$ and $\mu > 0$ such that*

$$\begin{pmatrix} A^\top K + KA + \lambda K & KB \\ B^\top K & -\mu I \end{pmatrix} \prec 0; \qquad \begin{pmatrix} \lambda K & 0 & C^\top \\ 0 & (\gamma - \mu)I & D^\top \\ C & D & \gamma I \end{pmatrix} \succ 0 \qquad (3.3.11)$$

then the peak-to-peak (or $\mathscr{L}_\infty$ induced) norm of the system is smaller than γ, i.e., $\|T\|_{\infty,\infty} < \gamma$.

Proof. The first inequality in (3.3.11) implies that

$$\frac{d}{dt} x(t)^\top K x(t) + \lambda x(t) K x(t) - \mu w(t)^\top w(t) \; < \; 0.$$

for all w and x for which $\dot{x} = Ax + Bw$. Now assume that $x(0) = 0$ and $w \in \mathscr{L}_\infty$ with $\|w\|_\infty \leq 1$. Then, since $K \succ 0$, we obtain (pointwise in $t \geq 0$) that

$$x(t)^\top K x(t) \leq \frac{\mu}{\lambda}.$$

Taking a Schur complement of the second inequality in (3.3.11) yields that

$$\begin{pmatrix} \lambda K & 0 \\ 0 & (\gamma - \mu)I \end{pmatrix} - \frac{1}{\gamma - \varepsilon} \begin{pmatrix} C & D \end{pmatrix}^\top \begin{pmatrix} C & D \end{pmatrix} \succ 0$$

so that, pointwise in $t \geq 0$ and for all $\|w\|_\infty \leq 1$, we can write

$$\langle z(t), z(t) \rangle \leq (\gamma - \varepsilon)[\lambda x(t)^\top K x(t) + (\gamma - \mu)w(t)^\top w(t)]$$
$$\leq \gamma(\gamma - \varepsilon)$$

Consequently, the peak-to-peak gain of the system is smaller than γ.   ∎

Remark 3.17 We emphasize that Proposition 3.16 gives only a sufficient condition for an upperbound γ of the peak-to-peak gain of the system. The minimal $\gamma \geq 0$ for which the there exist $K \succ 0$, $\lambda > 0$ and $\mu \geq 0$ such that (3.3.11) is satisfied is usually only an upperbound of the real peak-to-peak gain of the system.

3.4 Further reading

The amount of literature on Lyapunov theory is quite vast. We refer to [13, 25, 41, 63] for a number of standard works. Classical and extensive treatments of Lyapunov stability results can be found in [9] and [11]. For more details on the generalized H_2 norm we refer to [40]. A first variation of Theorem 3.8 appeared in [7].

3.5 Exercises

Exercise 1
Consider the 2 dimensional non-linear system

$$\dot{x} = x - y - x(x^2 + y^2) + \frac{xy}{\sqrt{x^2 + y^2}}$$

$$\dot{y} = x + y - y(x^2 + y^2) - \frac{x^2}{\sqrt{x^2 + y^2}}$$

(a) Verify that $(1,0)$ is a fixed point of this system

(b) Simulate a number of trajectories of the system with initial condition nearby the fixed point $(1,0)$.

(c) Try to make conclusions on the stability of the fixed point. Is it attractive, Lyapunov stable, asymptotically stable, unstable?

Exercise 2
Show that

(a) the quadratic function $V(x) := x^\top K x$ is positive definite if and only if K is a positive definite matrix.

(b) the function $V : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ defined as $V(x,t) := e^{-t}x^2$ is decrescent but not positive definite.

Exercise 3
A pendulum of mass m is connected to a servo motor which is driven by a voltage u. The angle which the pendulum makes with respect to the upright vertical axis through the center of rotation is denoted by θ (that is, $\theta = 0$ means that the pendulum is in upright position). The system is described by the equations

$$\begin{cases} J\frac{d^2\theta}{dt^2} = mlg\sin(\theta) + u \\ y = \theta \end{cases}$$

where l denotes the distance from the axis of the servo motor to the center of mass of the pendulum, J is the inertia and g is the gravitation constant. The system is specified by the constants $J = 0.03$, $m = 1$, $l = 0.15$ and $g = 10$.

(a) Determine the equilibrium points of this system.

(b) Are the equilibrium points Lyapunov stable? If so, determine a Lyapunov function.

(c) Linearize the system around the equilibrium points and provide a state space representation of the linearized systems.

(d) Verify whether the linearized systems are stable, unstable or asymptotically stable.

(e) A *proportional feedback* controller is a controller of the form $u = ky$ where $k \in \mathbb{R}$. Does there exists a proportional feedback controller such that the unstable equilibrium point of the system becomes asymptotically stable?

Exercise 4

Let a stability region $\mathbb{C}_{\text{stab}}$ be defined as those complex numbers $s \in \mathbb{C}$ which satisfy

$$\begin{cases} \text{Re}(s) < -\alpha & \text{and} \\ |s - c| < r & \text{and} \\ |\text{Im}(s)| < |\text{Re}(s)|. \end{cases}$$

where $\alpha > 0$, $c > 0$ and $r > 0$. Specify a real symmetric matrix $P \in \mathbb{S}^{2m \times 2m}$ such that $\mathbb{C}_{\text{stab}}$ coincides with the LMI region L_P as specified in Definition (3.7).

Exercise 5

Let $0 \le \alpha \le \pi$ and consider the Lyapunov equation $A^\top X + XA + I = 0$ where

$$A = \begin{pmatrix} \sin(\alpha) & \cos(\alpha) \\ -\cos(\alpha) & \sin(\alpha) \end{pmatrix}$$

Show that the solution X of the Lyapunov equation diverges in the sense that $\det(X) \longrightarrow \infty$ whenever $\alpha \longrightarrow 0$.

Exercise 6

Consider the suspension system in Exercise 3 of Chapter 2. Recall that the variable q_0 represents the road profile.

(a) Consider the case where $f = 0$ and $q_0 = 0$ (thus no active force between chassis and axle and a 'flat' road characteristic). Verify whether this system is asymptotically stable.

(b) Again with $f = 0$ and $q_0 = 0$, determine a Lyapunov function $V : X \to \mathbb{R}$ of this system and show that its derivative is negative along solutions of the autonomous behavior of the system.

(c) Design your favorite road profile q_0 in MATLAB and simulate the response of the system to this road profile (the force f is kept 0). Plot the variables q_1 and q_2. What are your conclusions?

(d) Consider, with $f = 0$, the transfer function T mapping the road profile q_0 to the output $\text{col}(q_1, q_2)$ of the system. Determine the norms $\|T\|_{H_\infty}$ and $\|T\|_{H_2}$.

Exercise 7

Consider the system $\dot{x} = Ax + Bw$, $z = Cx + Dw$ with

$$A = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 4 & -3 \\ 1 & -3 & -1 & -3 \\ 0 & 4 & 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(a) Write a program to compute the H_∞ norm of this system.

(b) Write a program to compute the H_2 norm of this system.

(c) Write a program to compute the generalized H_2 norm of this system.

Exercise 8

Consider a batch chemical reactor with a constant volume V of liquids. Inside the reactor the series reaction

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$
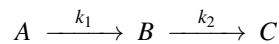
takes place. Here k_1 and k_2 represent the kinetic rate constants (1/sec.) for the conversions $A \to B$ and $B \to C$, respectively. The conversions are assumed to be irreversible which leads to the model equations

$$\dot{C}_A = -k_1 C_A$$
$$\dot{C}_B = k_1 C_A - k_2 C_B$$
$$\dot{C}_C = k_2 C_B$$

where C_A, C_B and C_C denote the concentrations of the components A, B and C, respectively, and k_1 and k_2 are positive constants. Reactant B is the desired product and we will be interested in the evolution of its concentration.

(a) Show that the system which describes the evolution of C_B is asymptotically stable.

(b) Determine a Lyapunov function for this system.

(c) Suppose that at time $t = 0$ the reactor is injected with an initial concentration $C_A(0) = 10$ (mol/liter) of reactant A and that $C_B(0) = C_C(0) = 0$. Plot the time evolution of the concentration C_B of reactant B if $(k_1, k_2) = (0.2, 0.4)$ and if $(k_1, k_2) = (0.3, 0.3)$.

Exercise 9

Consider the nonlinear scalar differential equation

$$\dot{x} = \sqrt{x}.$$

with initial condition $x(0) = 0$.

(a) Show that this differential equation does not satisfy the Lipschitz condition to guarantee uniqueness of solutions $x : \mathbb{R}_+ \to \mathbb{R}$.

(b) Show that the differential equation has at least two solutions $x(t)$, $t \geq 0$, with $x(0) = 0$.

Exercise 10

Consider the discrete time system

$$x(t+1) = f(x(t)). \tag{3.5.1}$$

(a) Show that x^* is an equilibrium point of (3.5.1) if and only if $f(x^*) = x^*$.

(b) Let $x(t+1) = Ax(t)$ be the linearization of (3.5.1) around the equilibrium point x^*. Derive an LMI feasibility test which is necessary and sufficient for A to have its eigenvalues in $\{z \in \mathbb{C} \mid |z| < 1\}$.

Exercise 11

Let $T(s) = C(Is-A)^{-1}B + D$ with A Hurwitz and $C \in \mathbb{R}^{1 \times n}$ (i.e., T is a single output stable transfer function). Let $\|T\|_{H_2}$ and $\|T\|_{2,\infty}$ denote the H_2 and the generalized H_2 norm of T, respectively. Show that

$$\|T\|_{H_2} = \|T\|_{2,\infty}.$$

Exercise 12

Given is the transfer function

$$T(s) = \frac{3(s-1)(s+0.5)}{(s+0.2)(s^2+0.8s+1.16)}.$$

Perform an LMI feasibility test to determine the smallest possible $\gamma > 0$ that bounds the generalized H_2 norm of T in the sense that

$$\|T\|_{2,\infty} := \sup_{0 < \|w\|_2 < \infty} \frac{\|z\|_\infty}{\|w\|_2} \leq \gamma.$$

Exercise 13

In chemical process industry, distillation columns play a key role to split an input stream of chemical species into two or more output streams of desired chemical species. Distillation is usually the most economical method for separating liquids, and consists of a process of multi-stage equilibrium separations. Figure 3.1 illustrates a typical distillation column.

Separation of input components, the *feed*, is achieved by controlling the transfer of components between the various *stages* (also called *trays* or *plates*), within the column, so as to produce output products at the bottom and at the top of the column. In a typical distillation system, two recycle streams are returned to the column. A *condenser* is added at the top of the column and a fraction of the overhead vapor V is condensed to form a liquid recycle L. The liquid recycle provides the liquid stream needed in the tower. The remaining fraction of V, is the *distillate-* or *top product*. A vaporizer or *reboiler* is added to the bottom of the column and a portion of the bottom liquid, L_b, is vaporized and recycled to the tower as a vapor stream V_b. This provides the vapor stream needed in the tower, while the remaining portion of L_b is the *bottom product*.

The column consists of n *stages*, numbered from top to bottom. The *feed* enters the column at stage n_f, with $1 < n_f < n$. The *feed flow*, F [kmol/hr], is a saturated liquid with composition z_F [mole fraction]. L [kmole/hr] denotes the *reflux flow rate* of the condenser, V_b [kmole/hr] is the *boilup flow rate* of the reboiler. The variable $w = \text{col}(L, V_b, F)$ is taken as input of the plant. The top product consists of a *distillate stream* D [kmol/hr], with composition X_d [mole fraction]. Likewise,
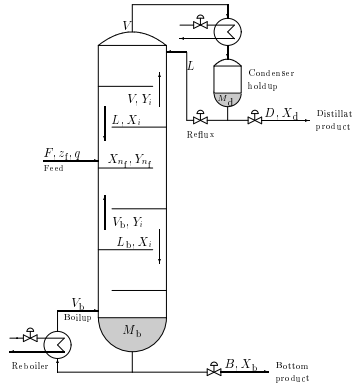
Figure 3.1: Binary distillation column

the bottom product consists of a *bottom stream B*, with composition X_B [mole fraction]. The output of the system is taken to be $z = \text{col}(X_d, X_b)$ and therefore consists of the distillate composition and bottom composition, respectively.

A model for this type of reactors is obtained as follows. The stages above the feed stage (index $i < n_f$) define the *enriching section* and those below the feed stage (index $i > n_f$) the *stripping section* of the column. The liquid flow rate in the stripping section is defined as $L_b = L + qF$ where $0 \leq q \leq 1$ is a constant. The vapor flow rate in the enriching section is given by $V = V_b + (1 - q)F$. The distillate and bottom product flow rates are $D = V - L$ and $B = L_b - V_b$, respectively. Denote by X_i and Y_i [mole fraction] the liquid and vapor compositions of stage i, respectively. For constant liquid holdup conditions, the *material balances* of the column are given as follows.

$$M_d \frac{dX_1}{dt} = VY_2 - (L+D)X_1 \qquad \text{Condenser stage}$$

$$M \frac{dX_i}{dt} = L(X_{i-1} - X_i) + V(Y_{i+1} - Y_i) \qquad 1 < i < n_f$$

$$M \frac{dX_f}{dt} = LX_{f-1} - L_bX_f + V_bY_{f+1} - VY_f + Fz_f \qquad \text{Feed stage}$$

$$M \frac{dX_i}{dt} = L_b(X_{i-1} - X_i) + V_b(Y_{i+1} - Y_i) \qquad n_f < i < n$$

$$M_b \frac{dX_n}{dt} = L_bX_{n-1} - V_bY_n - BX_n. \qquad \text{Reboiler stage}$$

Here, M, M_d and M_b [kmol] denote the nominal stage hold-up of material at the trays, the condenser and the bottom, respectively. The vapor-liquid equilibrium describes the relation between the vapor and liquid compositions Y_i and X_i on each stage i of the column and is given by the non-linear expression:

$$Y_i = \frac{aX_i}{1 + (a-1)X_i}, \qquad i = 1, \ldots, n.$$

where a is the so called *relative volatility* (dependent on the product). With x denoting the vector of components X_i, these equations yield a nonlinear model of the form

$$\dot{x} = f(x, w), \qquad z = g(x, w)$$

of state dimension n (equal to the number of stages).

We consider a medium-sized propane-butane distillation column whose physical parameters are given in Table 3.1. The last three entries in this table define w^*.

n	Number of stages	20
n_f	Feed stage	6
Md	Condenser holdup	200 [kmol]
Mb	Reboiler holdup	400 [kmol]
M	Stage holdup	50 [kmol]
z_f	Feed composition	0.5 [mole fraction]
q	Feed liquid fraction	1
a	Relative volatility	2.46
L^*	Reflux flow	1090 [kmol/hr]
V_b^*	Boilup vapor flow	1575 [kmol/hr]
F^*	Feed flow	1000 [kmol/hr]

Table 3.1: Operating point data

(a) Calculate an equilibrium state x^* of the model if the input is set to w^*. The equilibrium point (w^*, x^*, z^*) represents a steady-state or *nominal operating point* of the column.

(b) Construct (or compute) a linear model of the column when linearized around the equilibrium point (w^*, x^*, z^*).

(c) Is the linear model stable?

(d) Is the linear model asymptotically stable?

(e) Make a Bode diagram of the 2×3 transfer function of this system.

(f) Determine the responses of the system if the set-points of, respectively, L, V_b and F undergo a $+10\%$ step-change with respect to their nominal values L^*, V_b^* and F^*. Can you draw any conclusion on the (relative) sensitivity of the system with respect to its inputs w? Are there any major differences in the settling times?

Chapter 4

Controller synthesis

4.1 Introduction

In the previous chapter we have seen how various specifications such as stability and nominal performance can be analytically expressed as feasibility conditions on a suitable linear matrix inequality. This chapter is dedicated to the question how these specifications can actually be achieved by the design of a suitable feedback controller that is interconnected to a given plant as illustrated in Figure 4.1

In this chapter we provide a very powerful result that allows to step in a straightforward manner from performance *analysis* conditions, derived in the previous chapter as feasibility conditions of matrix inequalities, to the corresponding matrix inequalities for controller *synthesis*. This is achieved by a nonlinear and essentially bijective transformation of the controller parameters. We will introduce this transformation and discuss its consequences in the first section on single-objective controller synthesis. In fact, the main results of this section immediately provide a computational tool for
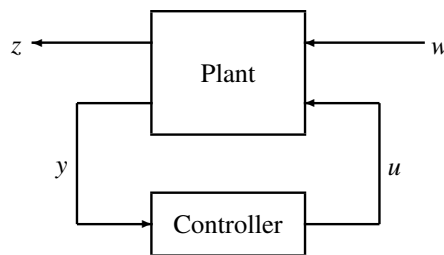

Figure 4.1: Closed-loop system

the synthesis of controllers that render closed loop systems asymptotically stable while achieving a quadratic performance specification on the controlled system. Some important special cases such as state feedback synthesis, full information control configurations and observer synthesis are obtained as special cases. In the remaining sections of this chapter we apply the synthesis results for multi-objective and mixed-objective controller synthesis problems.

4.2 Single-objective synthesis

4.2.1 The setup

With reference to Figure 4.1 we suppose that a linear time-invariant system (the plant) is given and described as

$$
\begin{pmatrix} \dot{x} \\ \hline z \\ y \end{pmatrix} = \begin{pmatrix} A & B_1 & B \\ \hline C_1 & D_1 & E \\ C & F & 0 \end{pmatrix} \begin{pmatrix} x \\ \hline w \\ u \end{pmatrix} \tag{4.2.1}
$$

where u is the control input, y the measured output available for control, w is a disturbance input and z is the controlled output. A *controller* is any finite dimensional linear time-invariant system described as

$$
\begin{pmatrix} \dot{x}_c \\ u \end{pmatrix} = \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} \begin{pmatrix} x_c \\ y \end{pmatrix} \tag{4.2.2}
$$

that has y as its input and produces the control u as its output. Controllers are hence simply parameterized by the matrices A_c, B_c, C_c and D_c in which the dimensions of the matrices are compatible with the dimensions of u and y. In particular, the dimension of the state vector x_c of the controller is not decided upon in advance.

The controlled or closed-loop system then admits the description

$$
\begin{pmatrix} \dot{\xi} \\ z \end{pmatrix} = \begin{pmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{pmatrix} \begin{pmatrix} \xi \\ w \end{pmatrix} \tag{4.2.3}
$$

where, after some straightforward algebra, one derives that

$$
\left(\begin{array}{c|c} \mathscr{A} & \mathscr{B} \\ \hline \mathscr{C} & \mathscr{D} \end{array} \right) = \left(\begin{array}{cc|c} A+BD_cC & BC_c & B_1+BD_cF \\ B_cC & A_c & B_cF \\ \hline C_1+ED_cC & EC_c & D_1+ED_cF \end{array} \right).
$$

We further denote by

$$
\mathscr{T} = \mathscr{C}(Is-\mathscr{A})^{-1}\mathscr{B}+\mathscr{D}
$$

the transfer function matrix of the controlled system. Throughout this chapter, all matrices and functions associated with the controlled system will be denoted with calligraphic symbols. Occasionally, it simplifies some calculations if we use the equivalent representation

$$
\left(\begin{array}{c|c} \mathscr{A} & \mathscr{B} \\ \hline \mathscr{C} & \mathscr{D} \end{array} \right) = \left(\begin{array}{cc|c} A & 0 & B_1 \\ 0 & 0 & 0 \\ \hline C_1 & 0 & D_1 \end{array} \right) + \left(\begin{array}{cc} 0 & B \\ I & 0 \\ \hline 0 & E \end{array} \right) \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} \left(\begin{array}{cc|c} 0 & I & 0 \\ C & 0 & F \end{array} \right). \tag{4.2.4}
$$

From this notation it is immediate that the left-hand side is an affine function of the controller parameters A_c, B_c, C_c, D_c.

Some special cases are worth mentioning:

- **Static state feedback** is the case where the controller has full access to the plant states and is memoryless in the sense that it has a zero dimensional state vector. Thus $C = I$, $F = 0$ and

$$\left(\begin{array}{cc} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{array} \right) = \left(\begin{array}{cc} A + BD_c & B_1 \\ C_1 + ED_c & D_1 \end{array} \right).$$

- **Dynamic state feedback** is the case where the controller has access to the states, i.e., $C = I$, $F = 0$. This gives

$$\left(\begin{array}{c|c} \mathscr{A} & \mathscr{B} \\ \hline \mathscr{C} & \mathscr{D} \end{array} \right) = \left(\begin{array}{cc|c} A + BD_c & BC_c & B_1 \\ B_c & A_c & 0 \\ \hline C_1 + ED_c C & EC_c & D_1 \end{array} \right).$$

- **Full information feedback** is the case where the plant delivers both states and disturbances to the controller. Thus $C = \left(\begin{smallmatrix} I \\ 0 \end{smallmatrix} \right)$, $F = \left(\begin{smallmatrix} 0 \\ I \end{smallmatrix} \right)$ and, under the additional assumption that the controller is memoryless,

$$\left(\begin{array}{cc} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{array} \right) = \left(\begin{array}{cc} A + BD_c^{\mathrm{fb}} & B_1 + BD_c^{\mathrm{ff}} \\ C_1 + ED_c^{\mathrm{fb}} & D_1 + ED_c^{\mathrm{ff}} \end{array} \right).$$

where we partitioned $D_c = \left(\begin{array}{cc} D_c^{\mathrm{fb}} & D_c^{\mathrm{ff}} \end{array} \right)$ in a feedback and a feedforward component.

4.2.2 From analysis to synthesis - a general procedure

As a paradigm example let us consider the design of a controller that achieves stability and quadratic performance for the controlled system (4.2.3). For that purpose, suppose that we have given a performance index

$$P = \left(\begin{array}{cc} Q & S \\ S^\top & R \end{array} \right) \quad \text{with } R \succcurlyeq 0. \tag{4.2.5}$$

In Proposition 3.9 of Chapter 3 we have revealed that a controller (4.2.2) renders (4.2.3) internally stable and establishes the nominal quadratic performance

$$\int_0^\infty \left(\begin{array}{c} w(t) \\ z(t) \end{array} \right)^\top \left(\begin{array}{cc} Q & S \\ S^\top & R \end{array} \right) \left(\begin{array}{c} w(t) \\ z(t) \end{array} \right) \mathrm{d}t \ \leq\ -\varepsilon^2 \int_0^\infty w(t)^\top w(t) \mathrm{d}t$$

for some $\varepsilon > 0$ if and only if

$$\sigma(\mathscr{A}) \subset \mathbb{C}^- \quad \text{and} \quad \left(\begin{array}{c} I \\ \mathscr{T}(i\omega) \end{array} \right)^* P \left(\begin{array}{c} I \\ \mathscr{T}(i\omega) \end{array} \right) \prec 0 \quad \text{for all} \ \ \omega \in \mathbb{R} \cup \{\infty\}$$

if and only if there exists a symmetric matrix $\mathscr{X}$ satisfying

$$
\left.
\begin{array}{r}
\mathscr{X} \;\succ\; 0 \\[1em]
\left(\begin{array}{cc}
I & 0 \\
0 & I \\
\hline
\mathscr{X}\mathscr{A} & \mathscr{X}\mathscr{B} \\
\mathscr{C} & \mathscr{D}
\end{array}\right)^{\top}
\left(\begin{array}{cc|cc}
0 & 0 & I & 0 \\
0 & Q & 0 & S \\
\hline
I & 0 & 0 & 0 \\
0 & S^{\top} & 0 & R
\end{array}\right)
\left(\begin{array}{cc}
I & 0 \\
0 & I \\
\hline
\mathscr{X}\mathscr{A} & \mathscr{X}\mathscr{B} \\
\mathscr{C} & \mathscr{D}
\end{array}\right)
\;\prec\; 0
\end{array}
\right\}
\tag{4.2.6}
$$

Recall that the second inequality in (4.2.6) can also be written as

$$
\left(\begin{array}{cc}
I & 0 \\
\hline
\mathscr{A} & \mathscr{B} \\
0 & I \\
\mathscr{C} & \mathscr{D}
\end{array}\right)^{\top}
\left(\begin{array}{cc|cc}
0 & \mathscr{X} & 0 & 0 \\
\mathscr{X} & 0 & 0 & 0 \\
\hline
0 & 0 & Q & S \\
0 & 0 & S^{\top} & R
\end{array}\right)
\left(\begin{array}{cc}
I & 0 \\
\hline
\mathscr{A} & \mathscr{B} \\
0 & I \\
\mathscr{C} & \mathscr{D}
\end{array}\right) =
$$

$$
=\left(\begin{array}{cc}
\mathscr{A}^{\top}\mathscr{X}+\mathscr{X}\mathscr{A} & \mathscr{X}\mathscr{B} \\
\mathscr{B}^{\top}\mathscr{X} & 0
\end{array}\right)
+\left(\begin{array}{cc}
0 & I \\
\mathscr{C} & \mathscr{D}
\end{array}\right)^{\top}
\left(\begin{array}{cc}
Q & S \\
S^{\top} & R
\end{array}\right)
\left(\begin{array}{cc}
0 & I \\
\mathscr{C} & \mathscr{D}
\end{array}\right) =
$$

$$
=\left(\begin{array}{cc}
I & 0 \\
\mathscr{X}\mathscr{A} & \mathscr{X}\mathscr{B} \\
\hline
0 & I \\
\mathscr{C} & \mathscr{D}
\end{array}\right)^{\top}
\left(\begin{array}{cc|cc}
0 & I & 0 & 0 \\
I & 0 & 0 & 0 \\
\hline
0 & 0 & Q & S \\
0 & 0 & S^{\top} & R
\end{array}\right)
\left(\begin{array}{cc}
I & 0 \\
\mathscr{X}\mathscr{A} & \mathscr{X}\mathscr{B} \\
\hline
0 & I \\
\mathscr{C} & \mathscr{D}
\end{array}\right) =
$$

$$
\prec 0.
$$

The corresponding *quadratic performance synthesis problem* amounts to finding controller parameters A_c, B_c, C_c, D_c and a symmetric $\mathscr{X}$ such that (4.2.6) holds.

Obviously, $\mathscr{A}$ depends on the controller parameters. Since $\mathscr{X}$ is also a decision variable, it follows that $\mathscr{X}\mathscr{A}$ depends, in general, non-linearly on the variables to be found. Hence, (4.2.6) is *not* an LMI in the decision variables $\mathscr{X}$ and A_c, B_c, C_c, D_c.

The essential idea will be to construct a transformation of the decision variables in such a manner that the transformed variables do enter linearly. Precisely, we will introduce a nonlinear mapping

$$
(\mathscr{X}, A_c, B_c, C_c, D_c) \quad \rightarrow \quad v := (X, Y, K, L, M, N)
\tag{4.2.7}
$$

that transforms the decision variables to a sixtuple of new decision variables v. In the transformed variables, we define the matrix functions

$$
\left.
\begin{array}{rcl}
\boldsymbol{X}(v) & := & \left(\begin{array}{cc} Y & I \\ I & X \end{array}\right) \\[1em]
\left(\begin{array}{cc} \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{array}\right) & := & \left(\begin{array}{cc|c} AY+BM & A+BNC & B_1+BNF \\ K & AX+LC & XB_1+LF \\ \hline C_1Y+EM & C_1+ENC & D_1+ENF \end{array}\right)
\end{array}
\right\}
\tag{4.2.8}
$$

that are affine in the variables v. In the sequel, we will construct a clever matrix $\mathscr{Y}$ such that the congruence transformations

$$
\mathscr{Y} \quad\text{and}\quad \left(\begin{array}{cc} \mathscr{Y} & 0 \\ 0 & I \end{array}\right),
\tag{4.2.9}
$$

applied to the inequalities (4.2.6), transform (4.2.6) into

$$\left. \begin{array}{rcl} \mathscr{Y}^\top \mathscr{X} \mathscr{Y} & \succ & 0 \\[4pt] \begin{pmatrix} I & 0 \\ 0 & I \\ \hline \mathscr{Y}^\top \mathscr{X} \mathscr{A} \mathscr{Y} & \mathscr{Y}^\top \mathscr{X} \mathscr{B} \\ \mathscr{C} \mathscr{Y} & \mathscr{D} \end{pmatrix}^\top \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & Q & 0 & S \\ \hline I & 0 & 0 & 0 \\ 0 & S^\top & 0 & R \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \\ \hline \mathscr{Y}^\top \mathscr{X} \mathscr{A} \mathscr{Y} & \mathscr{Y}^\top \mathscr{X} \mathscr{B} \\ \mathscr{C} \mathscr{Y} & \mathscr{D} \end{pmatrix} & \prec & 0 \end{array} \right\} \tag{4.2.10}$$

In this transformation, the blocks in (4.2.6) actually transform according to

$$\left. \begin{array}{rcl} \mathscr{Y}^\top \mathscr{X} \mathscr{Y} & = & \boldsymbol{X}(v) \\[4pt] \begin{pmatrix} \mathscr{Y} & 0 \\ 0 & I \end{pmatrix}^\top \begin{pmatrix} \mathscr{X} \mathscr{A} & \mathscr{X} \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{pmatrix} \begin{pmatrix} \mathscr{Y} & 0 \\ 0 & I \end{pmatrix} & = & \begin{pmatrix} \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{pmatrix} \end{array} \right\} \tag{4.2.11}$$

With the previous definitions of the matrix functions in v, it then follows that (4.2.10) is nothing but

$$\left. \begin{array}{rcl} \boldsymbol{X}(v) & \succ & 0 \\[4pt] \begin{pmatrix} I & 0 \\ 0 & I \\ \hline \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{pmatrix}^\top \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & Q & 0 & S \\ \hline I & 0 & 0 & 0 \\ 0 & S^\top & 0 & R \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \\ \hline \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{pmatrix} & \prec & 0. \end{array} \right\} \tag{4.2.12}$$

Therefore, and this is the essence, the original blocks that depend *non-linearly* on the decision variables $\mathscr{X}$ and A_c, B_c, C_c, D_c are now transformed into blocks that are *affine* functions of the new variables X, Y, K, L, M, N. Throughout, we will refer to (4.2.12) as the *synthesis inequalities*.

Moreover, if $\mathscr{Y}$ is nonsingular, the congruence transformations (4.2.9) yield that the inequalities (4.2.6) are, in fact, equivalent to (4.2.12).

For $R = 0$ (as it happens in the positive real performance index), the inequalities (4.2.12) are actually affine in v. For a general performance index with $R \succeq 0$, the second inequality in (4.2.12) is non-linear but convex in v. It is straightforward to transform it to a genuine LMI with a Schur complement argument. Since it is more convenient to stay with the inequalities in the form (4.2.12), we rather formulate a general auxiliary result that displays how to perform the linearization whenever it is required for computational purposes.

Lemma 4.1 (Linearization lemma) *Suppose that A and S are constant matrices, that $B(v)$, $Q(v) = Q(v)^\top$ depend affinely on a parameter v, and that $R(v)$ can be decomposed as $TU(v)^{-1}T^\top$ with $U(v)$ being affine. Then the non-linear matrix inequalities*

$$U(v) \succ 0, \quad \begin{pmatrix} A \\ B(v) \end{pmatrix}^\top \begin{pmatrix} Q(v) & S \\ S^\top & R(v) \end{pmatrix} \begin{pmatrix} A \\ B(v) \end{pmatrix} \prec 0$$

are equivalent to the linear matrix inequality

$$\begin{pmatrix} A^\top Q(v)A + A^\top SB(v) + B(v)^\top S^\top A & B(v)^\top T \\ T^\top B(v) & -U(v) \end{pmatrix} \prec 0.$$

In order to apply this lemma, consider the second inequality of (4.2.12):

$$
\begin{pmatrix} I & 0 \\ 0 & I \\ \hline \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{pmatrix}^{\top}
\begin{pmatrix} 0 & 0 & I & 0 \\ 0 & Q & 0 & S \\ \hline I & 0 & 0 & 0 \\ 0 & S^{\top} & 0 & R \end{pmatrix}
\begin{pmatrix} I & 0 \\ 0 & I \\ \hline \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{pmatrix} \prec 0.
\qquad (4.2.13)
$$

By taking $Q(v) = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}$, $U(v) = I$ and

$$
R(v) = \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix} = \begin{pmatrix} 0 \\ T \end{pmatrix} \begin{pmatrix} 0 & T^{\top} \end{pmatrix} \quad \text{where} \quad R = TT^{\top}
$$

is an arbitrary factorization of R, it then follows from Lemma 4.1 that (4.2.13) is equivalent to an LMI in the decision variables v, provided that $R \succcurlyeq 0$.

So far, the discussion in this section shows that the synthesis inequalities (4.2.12) are feasible in v if a controller (4.2.2) exists that achieves stability and quadratic performance for the controlled system (4.2.3). To also show the converse, let us suppose that the inequalities (4.2.12) are feasible. That is, suppose that we computed a solution v to (4.2.12). If we can find a pre-image $(\mathscr{X}, A_c, B_c, C_c, D_c)$ of v under the transformation (4.2.7) together with a nonsingular $\mathscr{Y}$ for which (4.2.11) holds, then we can simply reverse all the steps performed above to reveal that (4.2.12) is equivalent to (4.2.6). Therefore, the controller defined by A_c, B_c, C_c, D_c renders (4.2.6) satisfied and, hence, leads to the desired quadratic performance specification for the controlled system.

Before we comment on the resulting design procedure, let us first provide a proof of the following result that summarizes the discussion so far.

Theorem 4.2 *There exists a controller A_c, B_c, C_c, D_c and an $\mathscr{X}$ satisfying (4.2.6) if and only if there exists a v that solves the inequalities (4.2.12). If v satisfies (4.2.12), then $I - XY$ is nonsingular and there exist square nonsingular matrices U, V satisfying $I - XY = UV^{\top}$. With any such U and V,*

$$
\left.
\begin{aligned}
\mathscr{X} &= \begin{pmatrix} Y & V \\ I & 0 \end{pmatrix}^{-1} \begin{pmatrix} I & 0 \\ X & U \end{pmatrix} \\
\begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} &= \begin{pmatrix} U & XB \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} K - XAY & L \\ M & N \end{pmatrix} \begin{pmatrix} V^{\top} & 0 \\ CY & I \end{pmatrix}^{-1}
\end{aligned}
\right\}
\qquad (4.2.14)
$$

satisfy the LMI's (4.2.6) and therefore solves the quadratic synthesis problem.

Theorem 4.2 therefore provides an explicit solution to the synthesis problem that achieves stability and quadratic performance of the controlled system. In fact, the Lyapunov matrix $\mathscr{X}$ and the controller parameters A_c, B_c, C_c, D_c in (4.2.14) are the unique solutions of the equations

$$
\begin{pmatrix} Y & V \\ I & 0 \end{pmatrix} \mathscr{X} = \begin{pmatrix} I & 0 \\ X & U \end{pmatrix} \quad \text{and}
$$

$$
\begin{pmatrix} K & L \\ M & N \end{pmatrix} = \begin{pmatrix} U & XB \\ 0 & I \end{pmatrix} \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} \begin{pmatrix} V^{\top} & 0 \\ CY & I \end{pmatrix} + \begin{pmatrix} XAY & 0 \\ 0 & 0 \end{pmatrix}
\qquad (4.2.15)
$$

which do not involve matrix inversions. It is often preferred to directly solve the latter expressions by a numerically stable technique.

Proof. Suppose a controller and some $\mathscr{X}$ satisfy (4.2.6). Let us partition

$$\mathscr{X} = \left(\begin{array}{cc} X & U \\ U^\top & * \end{array} \right) \text{ and } \mathscr{X}^{-1} = \left(\begin{array}{cc} Y & V \\ V^\top & * \end{array} \right)$$

according to $\mathscr{A}$. Define

$$\mathscr{Y} = \left(\begin{array}{cc} Y & I \\ V^\top & 0 \end{array} \right) \text{ and } \mathscr{Z} = \left(\begin{array}{cc} I & 0 \\ X & U \end{array} \right) \text{ to get } \mathscr{Y}^\top \mathscr{X} = \mathscr{Z}. \tag{4.2.16}$$

Without loss of generality we can assume that the dimension of A_c is larger than that of A. Hence, U has more columns than rows, and we can perturb this block (since we work with strict inequalities) such that it has full row rank. Then $\mathscr{Z}$ has full row rank and, hence, $\mathscr{Y}$ has full column rank.

Due to $XY + UV^\top = I$, we infer that

$$\mathscr{Y}^\top \mathscr{X} \mathscr{Y} = \left(\begin{array}{cc} Y & I \\ I & X \end{array} \right) = \boldsymbol{X}(v).$$

This leads to the first relation in (4.2.11). Let us now consider

$$\left(\begin{array}{cc} \mathscr{Y} & 0 \\ 0 & I \end{array} \right)^\top \left(\begin{array}{cc} \mathscr{X}\mathscr{A} & \mathscr{X}\mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{array} \right) \left(\begin{array}{cc} \mathscr{Y} & 0 \\ 0 & I \end{array} \right) = \left(\begin{array}{cc} \mathscr{Y}^\top \mathscr{X}\mathscr{A}\mathscr{Y} & \mathscr{Y}^\top \mathscr{X}\mathscr{B} \\ \mathscr{C}\mathscr{Y} & \mathscr{D} \end{array} \right).$$

Using (4.2.4), a brief calculation reveals that

$$\left(\begin{array}{cc} \mathscr{Y}^\top \mathscr{X}\mathscr{A}\mathscr{Y} & \mathscr{Y}^\top \mathscr{X}\mathscr{B} \\ \mathscr{C}\mathscr{Y} & \mathscr{D} \end{array} \right) = \left(\begin{array}{cc} \mathscr{Z}\mathscr{A}\mathscr{Y} & \mathscr{Z}\mathscr{B} \\ \mathscr{C}\mathscr{Y} & \mathscr{D} \end{array} \right) = \left(\begin{array}{cc|c} AY & A & B_1 \\ 0 & XA & XB_1 \\ \hline C_1Y & C_1 & D_1 \end{array} \right) +$$

$$+ \left(\begin{array}{cc} 0 & B \\ I & 0 \\ \hline 0 & E \end{array} \right) \left[\left(\begin{array}{cc} U & XB \\ 0 & I \end{array} \right) \left(\begin{array}{cc} A_c & B_c \\ C_c & D_c \end{array} \right) \left(\begin{array}{cc} V^\top & 0 \\ CY & I \end{array} \right) + \left(\begin{array}{cc} XAY & 0 \\ 0 & 0 \end{array} \right) \right] \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & C & F \end{array} \right).$$

Introduce the new parameters K, L, M, N as in (4.2.15). We then infer

$$\left(\begin{array}{cc} \mathscr{Y}^\top \mathscr{X}\mathscr{A}\mathscr{Y} & \mathscr{Y}^\top \mathscr{X}\mathscr{B} \\ \mathscr{C}\mathscr{Y} & \mathscr{D} \end{array} \right) =$$

$$= \left(\begin{array}{cc|c} AY & A & B_1 \\ 0 & XA & XB_1 \\ \hline C_1Y & C_1 & D_1 \end{array} \right) + \left(\begin{array}{cc} 0 & B \\ I & 0 \\ \hline 0 & E \end{array} \right) \left(\begin{array}{cc} K & L \\ M & N \end{array} \right) \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & C & F \end{array} \right) =$$

$$= \left(\begin{array}{cc|c} AY + BM & A + BNC & B_1 + BNF \\ K & AX + LC & XB_1 + LF \\ \hline C_1Y + EM & C_1 + ENC & D_1 + ENF \end{array} \right) = \left(\begin{array}{cc} \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{array} \right).$$

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Hence the relations (4.2.11) are valid. Since $\mathscr{Y}$ has full column rank, (4.2.6) implies (4.2.10), and by (4.2.11), (4.2.10) is identical to (4.2.12). This proves necessity.

To reverse the arguments we assume that v is a solution of (4.2.12). Since $\boldsymbol{X}(v) \succ 0$, we infer that $I - XY$ is nonsingular. Hence we can factorize $I - XY = UV^\top$ with square and nonsingular U, V. Then $\mathscr{Y}$ and $\mathscr{Z}$ defined in (4.2.16) are, as well, square and nonsingular. Hence we can choose $\mathscr{X}$, A_c, B_c, C_c, D_c such that (4.2.15) hold true; this implies that, again, the relations (4.2.11) are valid. Therefore, (4.2.12) and (4.2.10) are identical. Since $\mathscr{Y}$ is nonsingular, a congruence transformation with $\mathscr{Y}^{-1}$ and $\mathrm{diag}(\mathscr{Y}^{-1}, I)$ leads from (4.2.10) back to (4.2.6) and the proof is finished. ∎

Theorem 4.2 has two important implications. Firstly, it provides a general procedure to derive from the analysis inequalities the corresponding synthesis inequalities. Secondly, it provides explicit expressions for the controller parameters A_c, B_c, C_c, D_c that achieve quadratic performance and a Lyapunov function $V(\xi) := \xi^\top \mathscr{X} \xi$ that proves stability of the controlled system (4.2.3). An explicit algorithm for the construction of such controllers is as follows:

Algorithm 4.3 (Synthesis algorithm) Aim: synthesize a controller (4.2.2) that solves the quadratic performance problem for the system (4.2.1).

> **Input:** The system (4.2.1) and the quadratic performance index (4.2.5).
>
> **Step 1:** Rewrite the analysis inequalities in the blocks $\mathscr{X}$, $\mathscr{X}\mathscr{A}$, $\mathscr{X}\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$ in order to be able to find a (formal) congruence transformation involving $\mathscr{Y}$ which leads to inequalities in the blocks $\mathscr{Y}^\top \mathscr{X} \mathscr{Y}$, $\mathscr{Y}^\top \mathscr{X} \mathscr{A} \mathscr{Y}$, $\mathscr{Y}^\top \mathscr{X} \mathscr{B}$, $\mathscr{C} \mathscr{Y}$, $\mathscr{D}$.
>
> **Step 2:** Perform the substitution (4.2.11) to arrive at matrix inequalities in the variables v.
>
> **Step 3:** Solve the synthesis inequalities (4.2.12) for v to obtain X, Y, K, L, M, N.
>
> **Step 4:** Determine non-singular matrices U and V such that $UV^\top = I - XY$
>
> **Output:** Define (4.2.14) as the controller parameters A_c, B_c, C_c and D_c and a Lyapunov matrix $\mathscr{X}$. These matrices satisfy the analysis inequalities (4.2.6).

The power of this procedure lies in its simplicity and its generality. Virtually all controller design methods that are based on matrix inequality analysis results can be converted with ease into the corresponding synthesis result. In the subsequent section we will include an extensive discussion of how to apply this technique to the various analysis results that have been obtained in the present notes.

Remark 4.4 (controller order) In the derivation of Theorem 4.2 we have not restricted the order of the controller. In proving necessity for the solvability of the synthesis inequalities, the size of A_c was arbitrary. The specific construction of a controller in proving sufficiency led to an A_c that has *the same size* as A. Hence Theorem 4.2 gives as a side result that controllers of order larger than that of the plant offer no advantage over controllers that have the same order as the plant. The story is very different in reduced order control: Then the intention is to include an explicit constraint $\dim(A_c) \le k$

for some k that is smaller than the dimension of A. It is not very difficult to derive the corresponding synthesis inequalities; however, they include *rank constraints* that are hard, if not impossible, to treat by current optimization techniques. We will only briefly comment on a concrete result later.

Remark 4.5 (strictly proper controllers) Note that the direct feed-through of the controller D_c is actually not transformed; we simply have $D_c = N$. If we intend to design a strictly proper controller (i.e. $D_c = 0$), we can just set $N = 0$ to arrive at the corresponding synthesis inequalities. The construction of the other controller parameters remains the same. Clearly, the same holds if one wishes to impose an arbitrary more refined structural constraint on the direct feed-through term as long as it can be expressed in terms of LMI's.

Remark 4.6 (numerical aspects) After having verified the solvability of the synthesis inequalities, we recommend to take some precautions to improve the conditioning of the calculations to reconstruct the controller out of the decision variable v. In particular, one should avoid that the parameters v get too large, and that $I - XY$ is close to singular. This may render the controller computation ill-conditioned. We have observed good results with the following two-step procedure:

- Add to the feasibility inequalities the bounds

$$\|X\| < \alpha, \quad \|Y\| < \alpha, \quad \left\| \begin{pmatrix} K & L \\ M & N \end{pmatrix} \right\| < \alpha$$

  as extra constraints and minimize α. Note that these bounds are equivalently rewritten in LMI form as

$$X \prec \alpha I, \quad Y \prec \alpha I, \quad \left(\begin{array}{cc|cc} \alpha I & 0 & K & L \\ 0 & \alpha I & M & N \\ \hline K^\top & M^\top & \alpha I & 0 \\ L^\top & N^\top & 0 & \alpha I \end{array} \right) \succ 0.$$

  Hence they can be easily included in the feasibility test, and one can directly minimize α to compute the smallest bound α_*.

- In a second step, one adds to the feasibility inequalities and to the bounding inequalities for some enlarged but fixed $\alpha > \alpha_*$ the extra constraint

$$\begin{pmatrix} Y & \beta I \\ \beta I & X \end{pmatrix} \succ 0.$$

  Of course, the resulting LMI system is feasible for $\beta = 1$. One can hence maximize β and obtain a supremal value $\beta_* > 1$. The value β_* gives an indication of the conditioning of the controller reconstruction procedure. In fact, the extra inequality is equivalent to $X - \beta^2 Y^{-1} \succ 0$. Hence, maximizing β amounts to 'pushing X away from Y^{-1}'. Therefore, this step is expected to push the smallest singular value of $I - XY$ away from zero. The larger the smaller singular value of $I - XY$, the larger one can choose the smallest singular values of both U and V in the factorization $I - XY = UV^\top$. This improves the conditioning of U and V, and renders the calculation of the controller parameters more reliable.

In the remainder of this section we apply the general design strategy to a number of specific controller synthesis problems.

4.2.3 H_∞ **design**

Recall that the optimal value of the H_∞ control synthesis problem is defined as

$$\gamma^* = \inf_{A_c,B_c,C_c,D_c \text{ such that } \sigma(\mathscr{A})\subset\mathbb{C}^-} \|\mathscr{T}\|_\infty.$$

Hence, by definition, $\gamma > \gamma^*$ if and only if there exists a controller that achieves

$$\sigma(\mathscr{A}) \subset \mathbb{C}^- \text{ and } \|\mathscr{T}\|_\infty < \gamma.$$

As we have seen in the previous chapter, these two properties are equivalent to stability and quadratic performance for the index

$$P = \left(\begin{array}{cc} Q & S \\ S^\top & R \end{array} \right) = \left(\begin{array}{cc} -\gamma I & 0 \\ 0 & (\gamma I)^{-1} \end{array} \right).$$

The corresponding synthesis inequalities (4.2.12) are rewritten with Lemma 4.1 to

$$\boldsymbol{X}(v) \succ 0, \quad \left(\begin{array}{ccc} \boldsymbol{A}(v)^\top + \boldsymbol{A}(v) & \boldsymbol{B}(v) & \boldsymbol{C}(v)^\top \\ \boldsymbol{B}(v)^\top & -\gamma I & \boldsymbol{D}(v)^\top \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) & -\gamma I \end{array} \right) \prec 0.$$

Note that the the optimal H_∞ value γ^* is then just given by the minimal γ for which these inequalities are feasible; one can directly compute γ^* by a standard LMI algorithm.

For the controller reconstruction, one should improve the conditioning (as described in the previous section) by an additional LMI optimization. We recommend not to perform this step with the optimal value γ^* itself but with a slightly increased value $\gamma > \gamma^*$. This is motivated by the observation that, at optimality, the matrix $\boldsymbol{X}(v)$ is often (but not always!) close to singular; then $I - XY$ is close to singular and it is expected to be difficult to render it better conditioned if γ is too close to the optimal value γ^*.

4.2.4 **Positive real design**

In this problem the goal is to test whether there exists a controller which renders the following two conditions satisfied:

$$\sigma(\mathscr{A}) \subset \mathbb{C}^-, \quad \mathscr{T}(i\omega)^* + \mathscr{T}(i\omega) \succ 0 \text{ for all } \omega \in \mathbb{R} \cup \{\infty\}.$$

This is equivalent to stability and quadratic performance for

$$P = \left(\begin{array}{cc} Q & S \\ S^\top & R \end{array} \right) = \left(\begin{array}{cc} 0 & -I \\ -I & 0 \end{array} \right),$$

and the corresponding synthesis inequalities read as

$$\boldsymbol{X}(v) \succ 0, \quad \left(\begin{array}{cc} \boldsymbol{A}(v)^\top + \boldsymbol{A}(v) & \boldsymbol{B}(v) - \boldsymbol{C}(v)^\top \\ \boldsymbol{B}(v)^\top - \boldsymbol{C}(v) & -\boldsymbol{D}(v) - \boldsymbol{D}(v)^\top \end{array} \right) \prec 0.$$

4.2.5 H_2 **design**

Let us define the linear functional

$$f(Z) := \text{trace}(Z).$$

Recall from Chapter 3 that $\mathscr{A}$ is stable and $\|\mathscr{T}\|_2 < \gamma$ if and only if there exists a symmetric $\mathscr{X}$ with

$$\mathscr{D} = 0, \ \ \mathscr{X} \succ 0, \ \ \begin{pmatrix} \mathscr{A}^\top \mathscr{X} + \mathscr{X} \mathscr{A} & \mathscr{X} \mathscr{B} \\ \mathscr{B}^\top \mathscr{X} & -\gamma I \end{pmatrix} \prec 0, \ \ f(\mathscr{C} \mathscr{X}^{-1} \mathscr{C}^\top) < \gamma. \tag{4.2.17}$$

The latter inequality is rendered affine in $\mathscr{X}$ and $\mathscr{C}$ by introducing the auxiliary variable (or slack variable) Z. Indeed, the analysis test is equivalent to

$$\mathscr{D} = 0, \ \ \begin{pmatrix} \mathscr{A}^\top \mathscr{X} + \mathscr{X} \mathscr{A} & \mathscr{X} \mathscr{B} \\ \mathscr{B}^\top \mathscr{X} & -\gamma I \end{pmatrix} \prec 0, \ \ \begin{pmatrix} \mathscr{X} & \mathscr{C}^\top \\ \mathscr{C} & Z \end{pmatrix} \succ 0, \ \ f(Z) < \gamma. \tag{4.2.18}$$

This version of the inequalities is suited to simply read-off the corresponding synthesis inequalities.

Corollary 4.7 *There exists a controller that renders* (4.2.18) *for some* $\mathscr{X}$, Z *satisfied if and only if there exist v and Z with*

$$\boldsymbol{D}(v) = 0, \ \ \begin{pmatrix} \boldsymbol{A}(v)^\top + \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{B}(v)^\top & -\gamma I \end{pmatrix} \prec 0, \ \ \begin{pmatrix} \boldsymbol{X}(v) & \boldsymbol{C}(v)^\top \\ \boldsymbol{C}(v) & Z \end{pmatrix} \succ 0, \ \ f(Z) < \gamma. \tag{4.2.19}$$

The proof of this statement and the controller construction are literally the same as for quadratic performance.

4.2.6 **Generalized** H_2 **design**

For the generalized H_2-norm $\|\mathscr{T}\|_{2g}$, we recall that $\mathscr{A}$ is stable and $\|\mathscr{T}\|_{2g} < \gamma$ if and only if

$$\mathscr{D} = 0, \ \ \mathscr{X} \succ 0, \ \ \begin{pmatrix} \mathscr{A}^\top \mathscr{X} + \mathscr{X} \mathscr{A} & \mathscr{X} \mathscr{B} \\ \mathscr{B}^\top \mathscr{X} & -\gamma I \end{pmatrix} \prec 0, \ \ \mathscr{C} \mathscr{X}^{-1} \mathscr{C}^\top \prec \gamma I.$$

These conditions are nothing but

$$\mathscr{D} = 0, \ \ \begin{pmatrix} \mathscr{A}^\top \mathscr{X} + \mathscr{X} \mathscr{A} & \mathscr{X} \mathscr{B} \\ \mathscr{B}^\top \mathscr{X} & -\gamma I \end{pmatrix} \prec 0, \ \ \begin{pmatrix} \mathscr{X} & \mathscr{C}^\top \\ \mathscr{C} & \gamma I \end{pmatrix} \succ 0$$

and it is straightforward to derive the synthesis LMI's.

Note that the corresponding inequalities are equivalent to (4.2.19) for the function

$$f(Z) = Z.$$

In contrast to the genuine H_2-problem, there is no need for the extra variable Z to render the inequalities affine.

Remark 4.8 A few generalizations and observations can be made on general H_2-type synthesis problems.

- If f assigns to Z its diagonal $\mathrm{diag}(z_1, \ldots, z_m)$ (where m is the dimension of Z), one characterizes a bound on the gain of $\mathscr{L}_2 \ni w \to z \in \mathscr{L}_\infty$ if equipping $\mathscr{L}_\infty$ with the norm $\|x\|_\infty :=$ $\mathrm{ess\,sup}_{t \geq 0} \max_k |x_k(t)|$ [40,43]. Note that the three concrete H_2-like analysis results for $f(Z) = \mathrm{trace}(Z)$, $f(Z) = Z$, $f(Z) = \mathrm{diag}(z_1, \ldots, z_m)$ are exact characterizations, and that the corresponding synthesis results do not involve any conservatism.

- In fact, Corollary 4.7 holds for any affine function f that maps symmetric matrices into symmetric matrices (of possibly different dimension) and that has the property $Z \succcurlyeq 0 \Rightarrow f(Z) \geq 0$. Hence, Corollary 4.7 admits many other specializations.

- Similarly as in the H_∞ problem, we can directly minimize the bound γ to find the optimal H_2-value or the optimal generalized H_2-value that can be achieved by stabilizing controllers.

- We observe that it causes no trouble in our general procedure to derive the synthesis inequalities if the underlying analysis inequalities involve certain auxiliary parameters (such as Z) as extra decision variables.

- It is instructive to equivalently rewrite (4.2.18) as $\mathscr{X} \succ 0$, $Z \succ 0$, $f(Z) < \gamma$ and

$$
\begin{pmatrix} I & 0 \\ \mathscr{X}\mathscr{A} & \mathscr{X}\mathscr{B} \\ \hline 0 & I \end{pmatrix}^\top \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & -\gamma I \end{pmatrix} \begin{pmatrix} I & 0 \\ \mathscr{X}\mathscr{A} & \mathscr{X}\mathscr{B} \\ \hline 0 & I \end{pmatrix} \prec 0,
$$

$$
\begin{pmatrix} I & 0 \\ \hline 0 & I \\ \mathscr{C} & \mathscr{D} \end{pmatrix}^\top \begin{pmatrix} -\mathscr{X} & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & Z^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ \hline 0 & I \\ \mathscr{C} & \mathscr{D} \end{pmatrix} \preccurlyeq 0.
$$

Note that the last inequality is non-strict and includes the algebraic constraint $\mathscr{D} = 0$. It can be equivalently replaced by

$$
\begin{pmatrix} I \\ \mathscr{C} \end{pmatrix}^\top \begin{pmatrix} -\mathscr{X} & 0 \\ 0 & Z^{-1} \end{pmatrix} \begin{pmatrix} I \\ \mathscr{C} \end{pmatrix} \prec 0, \quad \mathscr{D} = 0.
$$

The synthesis relations then read as $\boldsymbol{X}(v) \succ 0$, $Z \succ 0$, $f(Z) < \gamma$ and

$$
\begin{pmatrix} I & 0 \\ \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \hline 0 & I \end{pmatrix}^\top \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & -\gamma I \end{pmatrix} \begin{pmatrix} I & 0 \\ \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \hline 0 & I \end{pmatrix} \prec 0, \tag{4.2.20}
$$

$$
\begin{pmatrix} I \\ \boldsymbol{C}(v) \end{pmatrix}^\top \begin{pmatrix} -\boldsymbol{X}(v) & 0 \\ 0 & Z^{-1} \end{pmatrix} \begin{pmatrix} I \\ \boldsymbol{C}(v) \end{pmatrix} \prec 0, \quad \boldsymbol{D}(v) = 0. \tag{4.2.21}
$$

The first inequality is affine in v, whereas the second one can be rendered affine in v and Z with Lemma 4.1.

4.2.7 Design to bound the peak-to-peak norm

The controller (4.2.2) renders $\mathscr{A}$ stable and the bound

$$\|w\|_\infty \leq \gamma \|z\|_\infty \ \text{ for all } \ z \in \mathscr{L}_\infty$$

satisfied if (and not only if) there exist a symmetric $\mathscr{X}$ and real parameters λ, μ with

$$\lambda > 0, \ \begin{pmatrix} \mathscr{A}^\top \mathscr{X} + \mathscr{X} \mathscr{A} + \lambda \mathscr{X} & \mathscr{X} \mathscr{B} \\ \mathscr{B}^\top \mathscr{X} & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ \mathscr{C} & \mathscr{D} \end{pmatrix}^\top \begin{pmatrix} -\mu I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ \mathscr{C} & \mathscr{D} \end{pmatrix} \prec 0$$

$$\begin{pmatrix} 0 & I \\ \mathscr{C} & \mathscr{D} \end{pmatrix}^\top \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\gamma}I \end{pmatrix} \begin{pmatrix} 0 & I \\ \mathscr{C} & \mathscr{D} \end{pmatrix} \prec \begin{pmatrix} \lambda \mathscr{X} & 0 \\ 0 & (\gamma - \mu)I \end{pmatrix}.$$

(Note that $\mathscr{X} \succ 0$ is built in). The inequalities are obviously equivalent to

$$\lambda > 0, \ \begin{pmatrix} \mathscr{A}^\top \mathscr{X} + \mathscr{X} \mathscr{A} + \lambda \mathscr{X} & \mathscr{X} \mathscr{B} \\ \mathscr{B}^\top \mathscr{X} & -\mu I \end{pmatrix} \prec 0, \ \begin{pmatrix} \lambda \mathscr{X} & 0 & \mathscr{C}^\top \\ 0 & (\gamma - \mu)I & \mathscr{D}^\top \\ \mathscr{C} & \mathscr{D} & \gamma I \end{pmatrix} \succ 0,$$

and the corresponding synthesis inequalities thus read as

$$\lambda > 0, \ \begin{pmatrix} \boldsymbol{A}(v)^\top + \boldsymbol{A}(v) + \lambda \boldsymbol{X}(v) & \boldsymbol{B}(v) \\ \boldsymbol{B}(v)^\top & -\mu I \end{pmatrix} \prec 0, \ \begin{pmatrix} \lambda \boldsymbol{X}(v) & 0 & \boldsymbol{C}(v)^\top \\ 0 & (\gamma - \mu)I & \boldsymbol{D}(v)^\top \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) & \gamma I \end{pmatrix} \succ 0.$$

If these inequalities are feasible, one can construct a stabilizing controller which bounds the peak-to-peak norm of $z = \mathscr{T}w$ by γ. We would like to stress that the converse of this statement is not true since the analysis result involves conservatism.

Note that the synthesis inequalities are formulated in terms of the variables v, λ, and μ; hence they are *non-linear* since $\lambda \boldsymbol{X}(v)$ depends quadratically on λ and v. This problem can be overcome as follows: For fixed $\lambda > 0$, test whether the resulting *linear* matrix inequalities are feasible; if yes, one can stop since the bound γ on the peak-to-peak norm has been assured; if the LMI's are infeasible, one has to pick another $\lambda > 0$ and repeat the test.

In practice, it might be advantageous to find the best possible upper bound on the peak-to-peak norm that can be assured with the present analysis result. This would lead to the problem of minimizing γ under the synthesis inequality constraints as follows: Perform a line-search over $\lambda > 0$ to minimize $\gamma^*(\lambda)$, the minimal value of γ if $\lambda > 0$ is held fixed; note that the calculation of $\gamma^*(\lambda)$ indeed amounts to solving a genuine LMI problem. The line-search leads to the best achievable upper bound

$$\gamma^u = \inf_{\lambda > 0} \gamma^*(\lambda).$$

To estimate the conservatism, let us recall that the H_∞ norm $\|\mathscr{T}\|_\infty$ is a lower bound on the peak-to-peak norm of $\mathscr{T}$. If we calculate the minimal achievable H_∞-norm, say γ^l, of $\mathscr{T}$, we know that the actual optimal peak-to-peak gain must be contained in the interval
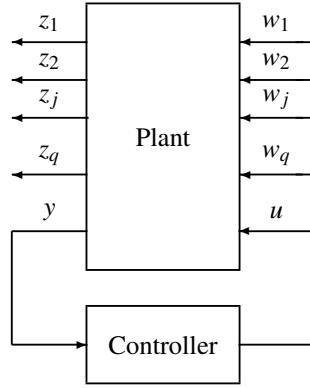
$$[\gamma^l, \gamma^u].$$

Figure 4.2: Multi-channel closed-loop system

If the length of this interval is small, we have a good estimate of the actual optimal peak-to-peak gain that is achievable by control, and if the interval is large, this estimate is poor.

4.3 Multi-objective and mixed controller design

In a realistic controller design problem one is usually not just confronted with a single-objective for the controlled system but one has to render various objectives satisfied simultaneously.

Consider, for this purpose the multi-channel control configuration as depicted in Figure 4.2 and suppose that the plant is a linear time-invariant multi-channel system described as

$$
\begin{pmatrix} \dot{x} \\ \hline z_1 \\ \vdots \\ z_q \\ y \end{pmatrix} = \left(\begin{array}{c|cccc} A & B_1 & \cdots & B_q & B \\ \hline C_1 & D_1 & \cdots & D_{1q} & E_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_q & D_{q1} & \cdots & D_q & E_q \\ C & F_1 & \cdots & F_q & 0 \end{array} \right) \begin{pmatrix} x \\ \hline w_1 \\ \vdots \\ w_q \\ u \end{pmatrix}.
\tag{4.3.1}
$$

where, as in (4.2.1), u denotes the control input, y the measured output, and $w_j \to z_j$ are the channels on which we want to impose certain robustness and/or performance objectives. Since we want to extend the design technique to mixed problems with various performance specifications on various channels, we already start at this point with a multi-channel system description. Collect the signals as

$$
z = \begin{pmatrix} z_1 \\ \vdots \\ z_q \end{pmatrix}, \qquad w = \begin{pmatrix} w_1 \\ \vdots \\ w_q \end{pmatrix}.
$$

We will not exclude the situation that some of the signals w_j or z_j are identical, but we consider an equal number of input- and output-signals. It might seem restrictive to only consider the diagonal channels and neglect the channels $w_j \to z_k$ for $j \neq k$. This is not the case. As a typical example, suppose we intend to impose for $z = \mathcal{T}w$ specifications on $L_j \mathcal{T} R_j$ where L_j, R_j are arbitrary matrices that pick out certain linear combinations of the signals z, w (or of the rows/columns of the transfer matrix if T is described by an LTI system). If we set $w = \sum_j R_j w_j$, $z_j = L_j z$, we are hence interested in specifications on the diagonal channels of

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_q \end{pmatrix} = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_q \end{pmatrix} \mathcal{T} \begin{pmatrix} R_1 & R_2 & \dots & R_q \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_q \end{pmatrix}.$$

If $\mathcal{T}$ is LTI, the selection matrices L_j and R_j can be easily incorporated into the realization to arrive at the description (4.3.1).

As before, a *controller* for (4.3.1) is a system of the form (4.2.2). The controlled or closed-loop system admits the description

$$\begin{pmatrix} \dot{\xi} \\ z \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} \xi \\ w \end{pmatrix} \text{ or } \begin{pmatrix} \dot{\xi} \\ z_1 \\ \vdots \\ z_q \end{pmatrix} = \left(\begin{array}{c|ccc} \mathcal{A} & \mathcal{B}_1 & \cdots & \mathcal{B}_q \\ \hline \mathcal{C}_1 & \mathcal{D}_1 & \cdots & \mathcal{D}_{1q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}_q & \mathcal{D}_{q1} & \cdots & \mathcal{D}_q \end{array} \right) \begin{pmatrix} \xi \\ w_1 \\ \vdots \\ w_q \end{pmatrix}. \quad (4.3.2)$$

where the corresponding input-output mappings (or transfer matrices) are denoted as

$$w = \mathcal{T}z \text{ or } \begin{pmatrix} z_1 \\ \vdots \\ z_q \end{pmatrix} = \begin{pmatrix} \mathcal{T}_1 & & * \\ & \ddots & \\ * & & \mathcal{T}_q \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_q \end{pmatrix}.$$

respectively, in which $\mathcal{T}_j$ admits the realization

$$\begin{pmatrix} \dot{\xi} \\ z_j \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{B}_j \\ \mathcal{C}_j & \mathcal{D}_j \end{pmatrix} \begin{pmatrix} \xi \\ w_j \end{pmatrix} \quad (4.3.3)$$

where

$$\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B}_j \\ \hline \mathcal{C}_j & \mathcal{D}_j \end{array} \right) = \left(\begin{array}{cc|c} A + BD_cC & BC_c & B_j + BD_cF_j \\ B_cC & A_c & B_cF_j \\ \hline C_j + E_jD_cC & E_jC_c & D_j + E_jD_cF_j \end{array} \right).$$

As a typical example, one might wish to keep the H_∞ norm of $z_1 = \mathcal{T}_1 w_1$ below a bound γ_1 to ensure robust stability against uncertainties entering as $w_1 = \Delta z_1$ where the stable mapping Δ has $\mathcal{L}_2$-gain smaller than $1/\gamma_1$, and render, at the same time, the H_2-norm of $z_2 = \mathcal{T}_2 w_2$ as small as possible to ensure good performance measured in the H_2-norm (such as guaranteeing small asymptotic variance of z_j against white noise inputs w_j or small energy of the output z_j against pulses as inputs w_j.)

Such a problem would lead to minimizing γ_2 over all controllers which render

$$\sigma(\mathscr{A}) \subset \mathbb{C}^-, \quad \|\mathscr{T}_1\|_\infty < \gamma_1, \quad \|\mathscr{T}_2\|_2 < \gamma_2 \tag{4.3.4}$$

satisfied. This is a *multi-objective H_2/H_∞ control problem* with two performance specifications.

Note that it is often interesting to investigate the trade-off between the H_∞-norm and the H_2-norm constraint. For that purpose one plots the curve of optimal values if varying γ_1 in some interval $[\gamma_1^l, \gamma_1^u]$ where the lower bound γ_1^l could be taken close to the smallest achievable H_∞-norm of $\mathscr{T}_1$. Note that the optimal value will be non-increasing if increasing γ_1. The actual curve will provide insight in how far one can improve performance by giving up robustness. In practice, it might be numerically advantageous to give up the hard constraints and proceed, alternatively, as follows: For fixed real weights α_1 and α_2, minimize the weighted sum

$$\alpha_1 \gamma_1 + \alpha_2 \gamma_2$$

over all controllers that satisfy (4.3.4). The larger α_j, the more weight is put on penalizing large values of γ_j, the more the optimization procedure is expected to reduce the corresponding bound γ_j.

Multi-objective control problems as formulated here are hard to solve. Let us briefly sketch one line of approach. The Youla parameterization reveals that the set of all $\mathscr{T}_j$ that can be obtained by internally stabilizing controllers can be parameterized as

$$T_1^j + T_2^j Q T_3^j \quad \text{with } Q \text{ varying freely in } RH_\infty^{p \times q}.$$

Here T_1^j, T_2^j, T_3^j are real-rational proper and stable transfer matrices which can be easily computed in terms of the system description (4.3.1) and an arbitrary stabilizing controller. Recall also that $RH_\infty^{p \times q}$ denotes the algebra of real-rational proper and stable transfer matrices of dimension $p \times q$. With this re-parameterization, the multi-objective control problem then amounts to finding a $Q \in RH_\infty^{p \times q}$ that minimizes γ_2 under the constraints

$$\|T_1^1 + T_2^1 Q T_3^1\|_\infty < \gamma_1, \quad \|T_1^2 + T_2^2 Q T_3^2\|_2 < \gamma_2. \tag{4.3.5}$$

After this re-formulation, we are hence faced with a convex optimization problem in the parameter Q which varies in the infinite-dimensional space RH_∞. A pretty standard Ritz-Galerkin approximation scheme leads to finite-dimensional problems. In fact, consider for a fixed real parameter $a > 0$ the sequence of finite-dimensional subspaces

$$\mathscr{S}_\nu := \left\{ Q_0 + Q_1 \frac{s-a}{s+a} + Q_2 \frac{(s-a)^2}{(s+a)^2} + \cdots + Q_\nu \frac{(s-a)^\nu}{(s+a)^\nu} : Q_0, \ldots, Q_\nu \in \mathbb{R}^{p \times q} \right\}$$

of the space $RH_\infty^{p \times q}$. Let us now denote the infimum of all γ_2 satisfying the constraint (4.3.5) for $Q \in RH_\infty^{p \times q}$ by γ_2^*, and that for $Q \in \mathscr{S}_\nu$ by $\gamma_2(\nu)$. Since $\mathscr{S}_\nu \subset RH_\infty^{p \times q}$, we clearly have

$$\gamma_2^* \leq \gamma_2(\nu+1) \leq \gamma_2(\nu) \quad \text{for all} \quad \nu = 0, 1, 2 \ldots.$$

Hence solving the optimization problems for increasing ν leads to a non-increasing sequence of values $\gamma(\nu)$ that are all upper bounds on the actual optimum γ_2^*. If we now note that any element of

Q can be approximated in the H_∞-norm with arbitrary accuracy by an element in $\mathscr{S}_\nu$ if ν is chosen sufficiently large, it is not surprising that $\gamma_2(\nu)$ actually converges to γ_2^* for $\nu \to \infty$. To be more precise, we need to assume that the strict constraint $\|T_1^1 + T_2^1 Q T_3^1\|_\infty < \gamma_1$ is feasible for $Q \in \mathscr{S}_\nu$ and some ν, and that T_1^2 and T_2^2 or T_2^3 are strictly proper such that $\|T_1^2 + T_2^2 Q T_3^2\|_2$ is finite for all $Q \in RH_\infty^{p \times q}$. Then it is not difficult to show that $\lim_{\nu \to \infty} \gamma_2(\nu) = \gamma_2^*$. Finally, we observe that computing $\gamma_2(\nu)$ is in fact an LMI problem.

The approach sketched above suffers from two severe disadvantages: First, if improving the approximation accuracy by letting ν grow, the size of the LMI's and the number of variables that are involved grow drastically what renders the corresponding computations slow. Second, increasing ν amounts to a potential increase of the McMillan degree of $Q \in \mathscr{S}_\nu$ what leads to controllers whose McMillan degree cannot be bounded a priori.

In view of these difficulties, it has been proposed to replace the multi-objective control problem by a *mixed control problem*. To prepare its definition, recall that the conditions (4.3.4) are guaranteed by the existence of symmetric matrices $\mathscr{X}_1$, $\mathscr{X}_2$, Z_2 satisfying

$$\mathscr{X}_1 \succ 0, \quad \begin{pmatrix} \mathscr{A}^\top \mathscr{X}_1 + \mathscr{X}_1 \mathscr{A} & \mathscr{X}_1 \mathscr{B}_1 & \mathscr{C}_1^\top \\ \mathscr{X}_1 \mathscr{B}_1 & -\gamma_1 I & \mathscr{D}_1^\top \\ \mathscr{C}_1 & \mathscr{D}_1 & -\gamma_1 I \end{pmatrix} \prec 0$$

$$\mathscr{D}_2 = 0, \quad \begin{pmatrix} \mathscr{A}^\top \mathscr{X}_2 + \mathscr{X}_2 \mathscr{A} & \mathscr{X}_2 \mathscr{B}_2 \\ \mathscr{B}_2^\top \mathscr{X}_2 & -\gamma_2 I \end{pmatrix} \prec 0, \quad \begin{pmatrix} \mathscr{X}_2 & \mathscr{C}_2^\top \\ \mathscr{C}_2 & Z_2 \end{pmatrix} \succ 0, \quad \operatorname{trace}(Z_2) < \gamma_2.$$

If trying to apply the general procedure to derive the synthesis inequalities, there is some trouble since the controller parameter transformation depends on the closed-loop Lyapunov matrix; here two such matrices $\mathscr{X}_1$, $\mathscr{X}_2$ do appear such that the technique breaks down. This observation itself motivates a remedy: Just force the two Lyapunov matrices to be equal. This certainly introduces conservatism that is, in general, hard to quantify. On the positive side, if one can find a common matrix

$$\mathscr{X} = \mathscr{X}_1 = \mathscr{X}_2$$

that satisfies the analysis relations, we can still guarantee (4.3.4) to hold. However, the converse is not true, since (4.3.4) does not imply the existence of common Lyapunov matrix to satisfy the above inequalities.

This discussion leads to the definition of the **mixed H_2/H_∞ control problem**: Minimize γ_2 subject to the existence of $\mathscr{X}$, Z_2 satisfying

$$\begin{pmatrix} \mathscr{A}^\top \mathscr{X} + \mathscr{X} \mathscr{A} & \mathscr{X} \mathscr{B}_1 & \mathscr{C}_1^\top \\ \mathscr{B}_1^\top \mathscr{X} & -\gamma_1 I & \mathscr{D}_1^\top \\ \mathscr{C}_1 & \mathscr{D}_1 & -\gamma_1 I \end{pmatrix} \prec 0$$

$$\mathscr{D}_2 = 0, \quad \begin{pmatrix} \mathscr{A}^\top \mathscr{X} + \mathscr{X} \mathscr{A} & \mathscr{X} \mathscr{B}_2 \\ \mathscr{B}_2^\top \mathscr{X} & -\gamma_2 I \end{pmatrix} \prec 0, \quad \begin{pmatrix} \mathscr{X} & \mathscr{C}_2^\top \\ \mathscr{C}_2 & Z_2 \end{pmatrix} \succ 0, \quad \operatorname{trace}(Z_2) < \gamma_2.$$

This problem is amenable to our general procedure. As before, one proves that the corresponding

synthesis LMI's are

$$
\begin{pmatrix}
\boldsymbol{A}(v)^\top + \boldsymbol{A}(v) & \boldsymbol{B}_1(v) & \boldsymbol{C}_1(v)^\top \\
\boldsymbol{B}_1(v)^\top & -\gamma_1 I & \boldsymbol{D}_1(v)^\top \\
\boldsymbol{C}_1(v) & \boldsymbol{D}_1(v) & -\gamma_1 I
\end{pmatrix} \prec 0
$$

$$
\boldsymbol{D}_2(v) = 0, \quad
\begin{pmatrix}
\boldsymbol{A}(v)^\top + \boldsymbol{A}(v) & \boldsymbol{B}_2(v) \\
\boldsymbol{B}_2(v)^\top & -\gamma_2 I
\end{pmatrix} \prec 0, \quad
\begin{pmatrix}
\boldsymbol{X}(v) & \boldsymbol{C}_2(v)^\top \\
\boldsymbol{C}_2(v) & Z_2
\end{pmatrix} \succ 0, \quad \text{trace}(Z_2) < \gamma_2,
$$

and the controller construction remains unchanged as in (4.2.14) in Theorem 4.2.

Let us conclude this section with some important remarks.

- After having solved the synthesis inequalities corresponding to the mixed problem for v and Z_2, one can construct a controller which satisfies (4.3.4) and which has a McMillan degree (size of A_c) that is not larger than (equal to) the size of A.

- For the controller resulting from mixed synthesis one can perform an analysis with different Lyapunov matrices $\mathscr{X}_1$ and $\mathscr{X}_2$ without any conservatism. In general, the actual H_∞-norm of $\mathscr{T}_1$ will be strictly smaller than γ_1, and the H_2-norm will be strictly smaller than the optimal value obtained from solving the mixed problem. Judging a mixed controller should, hence, rather be based on an additional non-conservative and direct analysis.

- Performing synthesis by searching for a common Lyapunov matrix introduces conservatism. Little is known about how to estimate this conservatism a priori. However, the optimal value of the mixed problem is always an upper bound of the optimal value of the actual multi-objective problem.

- Starting from a mixed controller, it has been suggested in [47, 48] how to compute sequences of upper and lower bounds, on the basis of solving LMI problems, that approach the actual optimal value. This allows to provide an a posteriori estimate of the conservatism that is introduced by setting $\mathscr{X}_1$ equal to $\mathscr{X}_2$.

- If starting from different versions of the analysis inequalities (e.g. through scaling the Lyapunov matrix), the artificial constraint $\mathscr{X}_1 = \mathscr{X}_2$ might lead to a different mixed control problem. Therefore, it is recommended to choose those analysis tests that are expected to lead to Lyapunov matrices which are close to each other. However, there is no general rule how to guarantee this property.

- In view of the previous remark, let us sketch one possibility to reduce the conservatism in mixed design. If we multiply the analysis inequalities for stability of $\mathscr{A}$ and for $\|\mathscr{T}_1\|_\infty < \gamma_1$ by an arbitrary real parameter $\alpha > 0$, we obtain

$$
\alpha \mathscr{X}_1 \succ 0, \quad
\begin{pmatrix}
\mathscr{A}^\top(\alpha \mathscr{X}_1) + (\alpha \mathscr{X}_1)\mathscr{A} & (\alpha \mathscr{X}_1)\mathscr{B}_1 & \alpha \mathscr{C}_1^\top \\
\mathscr{B}_1^\top(\alpha \mathscr{X}_1) & -\alpha \gamma_1 I & \alpha \mathscr{D}_1^\top \\
\alpha \mathscr{C}_1 & \alpha \mathscr{D}_1 & -\alpha \gamma_1 I
\end{pmatrix} \prec 0.
$$

If we multiply the last row and the last column of the second inequality with $\frac{1}{\alpha}$ (what is a congruence transformation) and if we introduce $\mathscr{Y}_1 := \alpha \mathscr{X}_1$, we arrive at the following equivalent version of the analysis inequality for the H_∞-norm constraint:

$$\mathscr{Y}_1 \succ 0, \quad \begin{pmatrix} \mathscr{A}^\top \mathscr{Y}_1 + \mathscr{Y}_1 \mathscr{A} & \mathscr{Y}_1 \mathscr{B}_1 & \mathscr{C}_1^\top \\ \mathscr{B}_1^\top \mathscr{Y}_1 & -\gamma_1 \alpha I & \mathscr{D}_1^\top \\ \mathscr{C}_1 & \mathscr{D}_1 & -\gamma_1 / \alpha I \end{pmatrix} \prec 0.$$

Performing mixed synthesis with this analysis inequality leads to optimal values of the mixed H_2 / H_∞ problem that depend on α. Each of these values form an upper bound on the actual optimal value of the multi-objective problem such that the best bound is found by performing a line-search over $\alpha > 0$.

- Contrary to previous approaches to the mixed problem, the one presented here does not require identical input- or output-signals of the H_∞ or H_2 channel. In view of their interpretation (uncertainty for H_∞ and performance for H_2), such a restriction is, in general, very unnatural. However, due to this flexibility, it is even more crucial to suitably scale the Lyapunov matrices.

- We can incorporate with ease various other performance or robustness specifications (formulated in terms of linear matrix inequalities) on other channels. Under the constraint of using for all desired specifications the same Lyapunov matrix, the design of a mixed controller is straightforward. Hence, one could conceivably consider a mixture of H_∞, H_2, generalized H_2, and peak-to-peak upper bound requirements on more than one channel. In its flexibility and generality, this approach is unique; however, one should never forget the conservatism that is involved.

- Using the same Lyapunov function might appear less restrictive if viewing the resulting procedure as a *Lyapunov shaping technique*. Indeed, one can start with the most important specification to be imposed on the controller. This amounts to solving a single-objective problem without conservatism. Then one keeps the already achieved property as a constraint and systematically imposes other specifications on other channels of the system to exploit possible additional freedom that is left in designing the controller. Hence, the Lyapunov function is shaped to realize additional specifications.

- Finally, constraints that are not necessarily related to input- output-specifications can be incorporated as well. As a nice example we mention the possibility to place the eigenvalues of $\mathscr{A}$ into an arbitrary LMI region $\{z : Q + Pz + P^\top \bar{z} \prec 0\}$. For that purpose one just has to include

$$\begin{pmatrix} p_{11} \boldsymbol{X}(v) + q_{11} \boldsymbol{A}(v) + q_{11} \boldsymbol{A}(v)^\top & \ldots & p_{1k} \boldsymbol{X}(v) + q_{1k} \boldsymbol{A}(v) + q_{k1} \boldsymbol{A}(v)^\top \\ \vdots & \ddots & \vdots \\ p_{k1} \boldsymbol{X}(v) + q_{k1} \boldsymbol{A}(v) + q_{1k} \boldsymbol{A}(v)^\top & \ldots & p_{kk} \boldsymbol{X}(v) + q_{kk} \boldsymbol{A}(v) + q_{kk} \boldsymbol{A}(v)^\top \end{pmatrix} \prec 0$$

in the set of synthesis LMI (see Chapter 2).

4.4   Elimination of parameters

The general procedure described in Section 4.2 leads to synthesis inequalities in the variables K, L, M, N and X, Y as well as some auxiliary variables. For specific problems it is often possible to eliminate some of these variables in order to reduce the computation time. For example, since K has the same size as A, eliminating K for a system with McMillan degree 20 would save 400 variables. In view of the fact that, in our experience, present-day solvers are practical for solving problems up to about one thousand variables, parameter elimination might be of paramount importance to be able to solve realistic design problems.

In general, one cannot eliminate any variable that appears in at least two synthesis inequalities. Hence, in mixed design problems, parameter elimination is typically only possible under specific circumstances. In single-objective design problems one has to distinguish various information structures. In output-feedback design problems, it is in general not possible to eliminate X, Y but it might be possible to eliminate some of the variables K, L, M, N if they only appear in one inequality. For example, in quadratic performance problems one can eliminate all the variables K, L, M, N. In state-feedback design, one can typically eliminate in addition X, and for estimation problems one can eliminate Y.

To understand which variables can be eliminated and how this is performed, we turn to a discussion of two topics that will be of relevance, namely the dualization of matrix inequalities and explicit solvability tests for specifically structured LMI's.

4.4.1   Dualization

The synthesis inequalities for quadratic performance can be written in the form (4.2.13). This inequality has the structure

$$\begin{pmatrix} I \\ M \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} I \\ M \end{pmatrix} \prec 0 \text{ and } R \succcurlyeq 0. \tag{4.4.1}$$

Let us re-formulate these conditions in geometric terms. Recall that (4.2.5) is partitioned according to

$$P = \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \in \mathbb{R}^{(k+l)\times(k+l)}$$

and observe that (4.4.1) is nothing but

$$P \prec 0 \text{ on } \operatorname{im}\begin{pmatrix} I \\ M \end{pmatrix} \quad \text{and} \quad P \succcurlyeq 0 \text{ on } \operatorname{im}\begin{pmatrix} 0 \\ I \end{pmatrix}.$$

Since the direct sum of $\operatorname{im}\begin{pmatrix} I \\ M \end{pmatrix}$ and $\operatorname{im}\begin{pmatrix} 0 \\ I \end{pmatrix}$ spans the whole $\mathbb{R}^{(k+l)\times(k+l)}$, we can apply the following dualization lemma if P is non-singular.

Lemma 4.9 (Dualization Lemma) *Let P be a non-singular symmetric matrix in $\mathbb{R}^{n \times n}$, and let $\mathscr{U}$, $\mathscr{V}$ be two complementary subspaces whose sum equals $\mathbb{R}^n$. Then*

$$x^\top P x < 0 \text{ for all } x \in \mathscr{U} \setminus \{0\} \quad and \quad x^\top P x \geq 0 \text{ for all } x \in \mathscr{V} \tag{4.4.2}$$

is equivalent to

$$x^\top P^{-1} x > 0 \text{ for all } x \in \mathscr{U}^\perp \setminus \{0\} \quad and \quad x^\top P^{-1} x \leq 0 \text{ for all } x \in \mathscr{V}^\perp. \tag{4.4.3}$$

Proof. Since $\mathscr{U} \oplus \mathscr{V} = \mathbb{R}^n$ is equivalent to $\mathscr{U}^\perp \oplus \mathscr{V}^\perp = \mathbb{R}^n$, it suffices to prove that (4.4.2) implies (4.4.3); the converse implication follows by symmetry. Let us assume that (4.4.2) is true. Moreover, let us assume that $\mathscr{U}$ and $\mathscr{V}$ have dimension k and l respectively. We infer from (4.4.2) that P has at least k negative eigenvalues and at least l non-negative eigenvalues. Since $k + l = n$ and since P is non-singular, we infer that P has exactly k negative and l positive eigenvalues. We first prove that P^{-1} is positive definite on $\mathscr{U}^\perp$. We assume, to the contrary, that there exists a vector $y \in \mathscr{U}^\perp \setminus \{0\}$ with $y^\top P^{-1} y \leq 0$. Define the non-zero vector $z = P^{-1} y$. Then z is not contained in $\mathscr{U}$ since, otherwise, we would conclude from (4.4.2) on the one hand $z^\top P z < 0$, and on the other hand $z \perp y = Pz$ what implies $z^\top P z = 0$. Therefore, the space $\mathscr{U}_e := \text{span}(z) + \mathscr{U}$ has dimension $k + 1$. Moreover, P is negative semi-definite on this space: for any $x \in \mathscr{U}$ we have

$$(z + x)^\top P(z + x) = y^\top P^{-1} y + y^\top x + x^\top y + x^\top P x = y^\top P^{-1} y + x^\top P x \leq 0.$$

This implies that P has at least $k + 1$ non-positive eigenvalues. Since 0 is not an eigenvalue of P, this contradicts the already established fact that P has exactly k negative eigenvalues.

Let us now prove that P^{-1} is negative semi-definite on $\mathscr{V}^\perp$. For that purpose we just observe that $P + \varepsilon I$ satisfies

$$x^\top (P + \varepsilon I) x < 0 \text{ for all } x \in \mathscr{U} \setminus \{0\} \quad and \quad x^\top (P + \varepsilon I) x > 0 \text{ for all } x \in \mathscr{V} \setminus \{0\}$$

for all small $\varepsilon > 0$. Due to what has been already proved, this implies

$$x^\top (P + \varepsilon I)^{-1} x > 0 \text{ for all } x \in \mathscr{U}^\perp \setminus \{0\} \text{ and } x^\top (P + \varepsilon I)^{-1} x < 0 \text{ for all } x \in \mathscr{V}^\perp \setminus \{0\}$$

for all small ε. Since P is non-singular, $(P + \varepsilon I)^{-1}$ converges to P^{-1} for $\varepsilon \to 0$. After taking the limit, we end up with

$$x^\top P^{-1} x \geq 0 \text{ for all } x \in \mathscr{U}^\perp \setminus \{0\} \quad and \quad x^\top P^{-1} x \leq 0 \text{ for all } x \in \mathscr{V}^\perp \setminus \{0\}.$$

Since we already know that the first inequality must be strict, the proof is finished. ∎

Based on the result of Lemma 4.9, we introduce

$$P^{-1} = \begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^\top & \tilde{R} \end{pmatrix} \in \mathbb{R}^{(k+l) \times (k+l)}$$

and observe that

$$\text{im} \begin{pmatrix} I \\ M \end{pmatrix}^\perp = \ker \begin{pmatrix} I & M^\top \end{pmatrix} = \text{im} \begin{pmatrix} -M^\top \\ I \end{pmatrix} \text{ and } \text{im} \begin{pmatrix} 0 \\ I \end{pmatrix}^\perp = \ker \begin{pmatrix} 0 & I \end{pmatrix} = \text{im} \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

Hence Lemma 4.9 implies that (4.4.1) is equivalent to

$$\begin{pmatrix} -M^\top \\ I \end{pmatrix}^\top \begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^\top & \tilde{R} \end{pmatrix} \begin{pmatrix} -M^\top \\ I \end{pmatrix} \succ 0 \ \text{ and } \ \tilde{Q} \preccurlyeq 0. \tag{4.4.4}$$

As an immediate consequence, we arrive at the following dual version of the quadratic performance synthesis inequalities that appeared in (4.2.13) and Theorem 4.2

Corollary 4.10 *Let* $P := \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix}$ *be non-singular and let* $P^{-1} := \begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^\top & \tilde{R} \end{pmatrix}$. *Then*

$$\left.\begin{pmatrix} I & 0 \\ 0 & I \\ \hline \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{pmatrix}^\top \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & Q & 0 & S \\ \hline I & 0 & 0 & 0 \\ 0 & S^\top & 0 & R \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \\ \hline \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{pmatrix} \begin{array}{c} \prec \ 0 \\ \\ R \ \succcurlyeq \ 0 \end{array}\right\}$$

is equivalent to

$$\left.\begin{pmatrix} -\boldsymbol{A}(v)^\top & -\boldsymbol{C}(v)^\top \\ -\boldsymbol{B}(v)^\top & -\boldsymbol{D}(v)^\top \\ \hline I & 0 \\ 0 & I \end{pmatrix}^\top \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & \tilde{Q} & 0 & \tilde{S} \\ \hline I & 0 & 0 & 0 \\ 0 & \tilde{S}^\top & 0 & \tilde{R} \end{pmatrix} \begin{pmatrix} -\boldsymbol{A}(v)^\top & -\boldsymbol{C}(v)^\top \\ -\boldsymbol{B}(v)^\top & -\boldsymbol{D}(v)^\top \\ \hline I & 0 \\ 0 & I \end{pmatrix} \begin{array}{c} \succ \ 0 \\ \\ \tilde{Q} \ \preccurlyeq \ 0 \end{array}\right\}$$

Remark 4.11 Recall that we require P to satisfy $R \succcurlyeq 0$ since, otherwise, the synthesis inequalities may not be convex. The above discussion reveals that any non-singular performance index has to satisfy $\tilde{Q} \preccurlyeq 0$ as well since, otherwise, we are sure that the synthesis inequalities are not feasible. We stress this point since, in general, $R \succcurlyeq 0$ does not imply $\tilde{Q} \preccurlyeq 0$. (Take e.g. $P \succ 0$ such that $P^{-1} \succ 0$.)

As an example, let us dualize the H_2 synthesis inequalities as formulated in (4.2.20)-(4.2.21).

Corollary 4.12 *For* $\gamma > 0$,

$$\begin{pmatrix} I & 0 \\ \hline \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \hline 0 & I \end{pmatrix}^\top \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ \hline 0 & 0 & -\gamma I \end{pmatrix} \begin{pmatrix} I & 0 \\ \hline \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \hline 0 & I \end{pmatrix} \prec 0$$

if and only if

$$\begin{pmatrix} -\boldsymbol{A}(v)^\top \\ -\boldsymbol{B}(v)^\top \\ \hline I \end{pmatrix}^\top \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ \hline 0 & 0 & -\frac{1}{\gamma}I \end{pmatrix} \begin{pmatrix} -\boldsymbol{A}(v)^\top \\ -\boldsymbol{B}(v)^\top \\ \hline I \end{pmatrix} \succ 0.$$

For $\boldsymbol{X}(v) \succ 0$ *and* $Z \succ 0$,

$$\begin{pmatrix} I \\ \boldsymbol{C}(v) \end{pmatrix}^\top \begin{pmatrix} -\boldsymbol{X}(v) & 0 \\ 0 & Z^{-1} \end{pmatrix} \begin{pmatrix} I \\ \boldsymbol{C}(v) \end{pmatrix} \prec 0$$

if and only if

$$\begin{pmatrix} -\boldsymbol{C}(v)^\top \\ I \end{pmatrix}^\top \begin{pmatrix} -\boldsymbol{X}(v)^{-1} & 0 \\ 0 & Z \end{pmatrix} \begin{pmatrix} -\boldsymbol{C}(v)^\top \\ I \end{pmatrix} \succ 0.$$

Again, the linearization Lemma 4.1 allows to render the first and the second dual inequalities affine in γ and $\boldsymbol{X}(v)$ respectively.

4.4.2 Projection

Let us now turn to specific linear matrix inequalities for which one can easily derive explicit solvability tests or which can be simplified considerably by eliminating variables. We start with a simple example that is cited for later reference.

Lemma 4.13 *The inequality*

$$\begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22}+X & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \prec 0$$

in the symmetric unknown X has a solution if and only if

$$\begin{pmatrix} P_{11} & P_{13} \\ P_{31} & P_{33} \end{pmatrix} \prec 0.$$

Proof. The 'only if' statement is obvious by canceling the second row/column. To prove the 'if part', observe that any X with

$$X \prec -P_{22} + \begin{pmatrix} P_{11} & P_{13} \end{pmatrix} \begin{pmatrix} P_{11} & P_{13} \\ P_{31} & P_{33} \end{pmatrix}^{-1} \begin{pmatrix} P_{12} \\ P_{32} \end{pmatrix} \prec 0$$

(such as $X = -\alpha I$ for sufficiently large $\alpha > 0$) is a solution (Schur). ∎

Remark. This result extends to finding a common solution to a whole system of LMI's, due to the following simple fact: For finitely many matrices $Q_1, \ldots, Q_m$, there always exists an X with $X \prec Q_j$, $j = 1, \ldots, m$.

The first of three more advanced results in this vain is just a simple consequence of a Schur complement argument and it can be viewed as a powerful variation of what is often called the technique of 'completing the squares'.

Lemma 4.14 (Projection Lemma) *Let P be a symmetric matrix partitioned into three rows/columns and consider the LMI*

$$\begin{pmatrix} P_{11} & P_{12}+X^\top & P_{13} \\ P_{21}+X & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \prec 0 \qquad (4.4.5)$$

in the unstructured matrix X. There exists a solution X of (4.4.5) *if and only if*

$$\begin{pmatrix} P_{11} & P_{13} \\ P_{31} & P_{33} \end{pmatrix} \prec 0 \ \ and \ \ \begin{pmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{pmatrix} \prec 0. \tag{4.4.6}$$

If (4.4.6) *hold, one particular solution is given by*

$$X = P_{32}^\top P_{33}^{-1} P_{31} - P_{21}. \tag{4.4.7}$$

Proof. (Only if) If (4.4.5) has a solution then (4.4.6) just follow from (4.4.5) by canceling the first or second block row/column.

(If) Suppose that (4.4.6) holds. Then $P_{33} \prec 0$ and we observe that (4.4.5) is equivalent to (Schur complement)

$$\begin{pmatrix} P_{11} & P_{12} + X^\top \\ P_{21} + X & P_{22} \end{pmatrix} - \begin{pmatrix} P_{13} \\ P_{23} \end{pmatrix} P_{33}^{-1} \begin{pmatrix} P_{31} & P_{32} \end{pmatrix} \prec 0.$$

Due to (4.4.6), the diagonal blocks are negative definite. X defined in (4.4.7) just renders the off-diagonal block zero such that it is a solution of the latter matrix inequality. ∎

An even more powerful generalization is the so-called projection lemma.

Lemma 4.15 (Projection Lemma) *For arbitrary A, B and a symmetric P, the LMI*

$$A^\top X B + B^\top X^\top A + P \prec 0 \tag{4.4.8}$$

in the unstructured X has a solution if and only if

$$Ax = 0 \ or \ Bx = 0 \ \ imply \ \ x^\top P x < 0 \ or \ x = 0. \tag{4.4.9}$$

If $A_\perp$ and $B_\perp$ denote arbitrary matrices whose columns form a basis of $\ker(A)$ *and* $\ker(B)$ *respectively,* (4.4.9) *is equivalent to*

$$A_\perp^\top P A_\perp \prec 0 \ \ and \ \ B_\perp^\top P B_\perp \prec 0. \tag{4.4.10}$$

We give a full proof of the Projection Lemma since it provides a scheme for constructing a solution X if it exists. It also reveals that, in suitable coordinates, Lemma 4.15 reduces to Lemma 4.14 if the kernels of A and B together span the whole space.

Proof. The proof of 'only if' is trivial. Indeed, let us assume that there exists some X with $A^\top X B + B^\top X^\top A + P \prec 0$. Then $Ax = 0$ or $Bx = 0$ with $x \neq 0$ imply the desired inequality $0 > x^\top (A^\top X B + B^\top X^\top A + P)x = x^\top P x$.

For proving 'if', let $S = (S_1 \ S_2 \ S_3 \ S_4)$ be a nonsingular matrix such that the columns of S_3 span $\ker(A) \cap \ker(B)$, those of $(S_1 \ S_3)$ span $\ker(A)$, and those of $(S_2 \ S_3)$ span $\ker(B)$. Instead of (4.4.8), we consider the equivalent inequality $S^\top(4.4.8)S < 0$ which reads as

$$(AS)^\top X (BS) + (BS)^\top X^\top (AS) + S^\top P S \prec 0. \tag{4.4.11}$$

Now note that AS and BS have the structure $(0\ A_2\ 0\ A_4)$ and $(B_1\ 0\ 0\ B_4)$ where $(A_2\ A_4)$ and $(B_1\ B_4)$ have full column rank respectively. The rank properties imply that the equation

$$(AS)^\top X(BS) = \begin{pmatrix} 0 \\ A_2^\top \\ 0 \\ A_4^\top \end{pmatrix} X \begin{pmatrix} B_1 & 0 & 0 & B_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ Z_{21} & 0 & 0 & Z_{24} \\ 0 & 0 & 0 & 0 \\ Z_{41} & 0 & 0 & Z_{44} \end{pmatrix}$$

has a solution X for arbitrary Z_{21}, Z_{24}, Z_{41}, Z_{44}. With $Q := S^\top PS$ partitioned accordingly, (4.4.11) hence reads as

$$\left(\begin{array}{ccc|c} Q_{11} & Q_{12}+Z_{21}^\top & Q_{13} & Q_{14}+Z_{41}^\top \\ Q_{21}+Z_{21} & Q_{22} & Q_{23} & Q_{24}+Z_{24} \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} \\ \hline Q_{41}+Z_{41} & Q_{42}+Z_{24}^\top & Q_{43} & Q_{44}+Z_{44}+Z_{44}^\top \end{array} \right) \prec 0 \qquad (4.4.12)$$

with *free* blocks Z_{21}, Z_{24}, Z_{41}, Z_{44}. Since

$$\ker(AS) = \operatorname{im} \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \ker(BS) = \operatorname{im} \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix},$$

the hypothesis (4.4.9) just amounts to the conditions

$$\begin{pmatrix} Q_{11} & Q_{13} \\ Q_{31} & Q_{33} \end{pmatrix} \prec 0 \quad \text{and} \quad \begin{pmatrix} Q_{22} & Q_{23} \\ Q_{32} & Q_{33} \end{pmatrix} \prec 0.$$

By Lemma 4.14, we can hence find a matrix Z_{21} which renders the marked 3×3 block in (4.4.12) negative definite. The blocks Z_{41} and Z_{24} can be taken arbitrary. After having fixed Z_{21}, Z_{41}, Z_{24}, we can choose Z_{44} according to Lemma 4.13 such that the whole matrix on the left-hand side of (4.4.12) is negative definite. ∎

Remark 4.16 We can, of course, replace $\prec$ everywhere by $\succ$. It is important to recall that the unknown X is unstructured. If one requires X to have a certain structure (such as being symmetric), the tests, if existing at all, are much more complicated. There is, however, a generally valid extension of the Projection Lemma to block-triangular unknowns X [42]. Note that the results do not hold true as formulated if just replacing the strict inequalities by non-strict inequalities (as it is sometimes erroneously claimed in the literature)! Again, it is possible to provide a full generalization of the Projection Lemma to non-strict inequalities.

4.4.3 Elimination

As in Subsection 4.4.1, let

$$P = \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \text{ with } R \succcurlyeq 0 \quad \text{and} \quad P^{-1} = \begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^\top & \tilde{R} \end{pmatrix} \text{ with } \tilde{Q} \preccurlyeq 0 \qquad (4.4.13)$$

and let us consider the quadratic inequality

$$\begin{pmatrix} I \\ A^\top X B + C \end{pmatrix}^\top P \begin{pmatrix} I \\ A^\top X B + C \end{pmatrix} \prec 0 \tag{4.4.14}$$

in the unstructured unknown X. According to Lemma 4.9, we can dualize this inequality to

$$\begin{pmatrix} -B^\top X^\top A - C^\top \\ I \end{pmatrix}^\top P^{-1} \begin{pmatrix} -B^\top X^\top A - C^\top \\ I \end{pmatrix} \succ 0. \tag{4.4.15}$$

It is pretty straightforward to derive necessary conditions for the solvability of (7.2.10). Indeed, let us assume that (7.2.10) holds for some X. If $A_\perp$ and $B_\perp$ denote basis matrices of $\ker(A)$ and $\ker(B)$ respectively, we infer

$$\begin{pmatrix} I \\ A^\top X B + C \end{pmatrix} B_\perp = \begin{pmatrix} I \\ C \end{pmatrix} B_\perp \quad \text{and} \quad \begin{pmatrix} -B^\top X^\top A - C' \\ I \end{pmatrix} A_\perp = \begin{pmatrix} -C' \\ I \end{pmatrix} A_\perp.$$

Since $B_\perp^\top (7.2.10) B_\perp \prec 0$ and $A_\perp^\top (4.4.15) A_\perp \succ 0$, it follows that the two inequalities

$$B_\perp^\top \begin{pmatrix} I \\ C \end{pmatrix}^\top P \begin{pmatrix} I \\ C \end{pmatrix} B_\perp \prec 0 \quad \text{and} \quad A_\perp^\top \begin{pmatrix} -C^\top \\ I \end{pmatrix}^\top P^{-1} \begin{pmatrix} -C^\top \\ I \end{pmatrix} A_\perp \succ 0 \tag{4.4.16}$$

are necessary conditions for a solution of (7.2.10) to exist. The inequalities (4.4.16) do no longer depend on the decision variable X and are therefore easily verified. Interestingly, the conditions (4.4.16) are also sufficient for the feasibility of (7.2.10).

Lemma 4.17 (Elimination Lemma) *Under the hypotheses* (4.4.13) *on P, the inequality* (7.2.10) *has a solution if and only if* (4.4.16) *hold true.*

Proof. It remains to prove that (4.4.16) implies the existence of a solution of (7.2.10). To see this, let us first reveal that one can assume without loss of generality that $R \succ 0$ and $\tilde{Q} \prec 0$. For that purpose we need to have information about the inertia of P. Since $R \succcurlyeq 0$, P and P^{-1} have $\text{size}(R)$ positive eigenvalues (since none of the eigenvalues can vanish). Similarly, $\tilde{Q} \preccurlyeq 0$ implies that P^{-1} and P have $\text{size}(\tilde{Q}) = \text{size}(Q)$ negative eigenvalues. Let us now consider (7.2.10) with the perturbed data

$$P_\varepsilon := \begin{pmatrix} Q & S \\ S^\top & R + \varepsilon I \end{pmatrix} \quad \text{where} \quad \varepsilon > 0$$

is fixed sufficiently small such that (4.4.16) persist to hold for P_ε, and such that P_ε and P have the same number of positive and negative eigenvalues. Trivially, the right-lower block of P_ε is positive definite. The Schur complement $Q - S(R + \varepsilon I)^{-1}S^\top$ of this right-lower block must be negative definite since P_ε has $\text{size}(Q)$ negative and $\text{size}(R)$ positive eigenvalues. Hence the left-upper block of P_ε^{-1} which equals $[Q - S(R + \varepsilon I)^{-1}S^\top]^{-1}$ is negative definite as well. If the result is proved with $R \succ 0$ and $\tilde{Q} \prec 0$, we can conclude that (7.2.10) has a solution X for the perturbed data P_ε. Since $P_0 \preccurlyeq P_\varepsilon$, the very same X also satisfies the original inequality for P_0.

Let us hence assume from now on that $R \succ 0$ and $\tilde{Q} \prec 0$. The left-hand side of (7.2.10) equals

$$\begin{pmatrix} I \\ C \end{pmatrix}^\top P \begin{pmatrix} I \\ C \end{pmatrix} + (A^\top X B)^\top (S^\top + RC) + (S^\top + RC)^\top (A^\top X B) + (A^\top X B)^\top R (A^\top X B).$$

Hence (7.2.10) is equivalent to (Schur)

$$\begin{pmatrix} \begin{pmatrix} I \\ C \end{pmatrix}^\top P \begin{pmatrix} I \\ C \end{pmatrix} + (A^\top X B)^\top (S^\top + RC) + (S^\top + RC)^\top (A^\top X B) & (A^\top X B)^\top \\ (A^\top X B) & -R^{-1} \end{pmatrix} \prec 0$$

or

$$\begin{pmatrix} \begin{pmatrix} I \\ C \end{pmatrix}^\top P \begin{pmatrix} I \\ C \end{pmatrix} & 0 \\ 0 & -R^{-1} \end{pmatrix} +$$

$$+ \begin{pmatrix} A(S^\top + RC)^\top \\ A \end{pmatrix}^\top X \begin{pmatrix} B & 0 \end{pmatrix} + \begin{pmatrix} B^\top \\ 0 \end{pmatrix} X^\top \begin{pmatrix} A(S^\top + RC) & A \end{pmatrix} \prec 0. \quad (4.4.17)$$

The inequality (4.4.17) has the structure as required in the Projection Lemma. We need to show that

$$\begin{pmatrix} B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0, \quad \begin{pmatrix} x \\ y \end{pmatrix} \neq 0 \qquad (4.4.18)$$

or

$$\begin{pmatrix} A(S^\top + RC) & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0, \quad \begin{pmatrix} x \\ y \end{pmatrix} \neq 0 \qquad (4.4.19)$$

imply

$$\begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} \begin{pmatrix} I \\ C \end{pmatrix}^\top P \begin{pmatrix} I \\ C \end{pmatrix} & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^\top \begin{pmatrix} I \\ C \end{pmatrix}^\top P \begin{pmatrix} I \\ C \end{pmatrix} x - y^\top y \prec 0. \quad (4.4.20)$$

In a first step we show that (4.4.17) and hence (7.2.10) have a solution if $A = I$. Let us assume (4.4.18). Then (4.4.20) is trivial if $x = 0$. For $x \neq 0$ we infer $Bx = 0$ and the first inequality in (4.4.16) implies

$$x^\top \begin{pmatrix} I \\ C \end{pmatrix}^\top P \begin{pmatrix} I \\ C \end{pmatrix} x < 0$$

what shows that (4.4.20) is true. Let us now assume (4.4.19) with $A = I$. We infer $x \neq 0$ and $y = -(S^\top + RC)x$. The left-hand side of (4.4.20) is nothing but

$$x^\top \begin{pmatrix} I \\ C \end{pmatrix}^\top P \begin{pmatrix} I \\ C \end{pmatrix} x - x^\top (S^\top + RC)^\top R^{-1} (S^\top + RC) x =$$

$$= x^\top \begin{pmatrix} I \\ C \end{pmatrix}^\top P \begin{pmatrix} I \\ C \end{pmatrix} x - x^\top \begin{pmatrix} I \\ C \end{pmatrix}^\top \begin{pmatrix} S \\ R \end{pmatrix} R^{-1} \begin{pmatrix} S^\top & R \end{pmatrix} \begin{pmatrix} I \\ C \end{pmatrix} x =$$

$$= x^\top \begin{pmatrix} I \\ C \end{pmatrix}^\top \left[P - \begin{pmatrix} SR^{-1}S^\top & S \\ S^\top & R \end{pmatrix} \right] \begin{pmatrix} I \\ C \end{pmatrix} x = x^\top (Q - SR^{-1}S^\top) x$$

what is indeed negative since $\tilde{Q}^{-1} = Q - SR^{-1}S^\top \prec 0$ and $x \neq 0$. We conclude that, for $A = I$, (4.4.17) and hence

$$\begin{pmatrix} I \\ XB+C \end{pmatrix}^\top P \begin{pmatrix} I \\ XB+C \end{pmatrix} \prec 0$$

have a solution.

By symmetry –since one can apply the arguments provided above to the dual inequality (4.4.15)– we can infer that

$$\begin{pmatrix} I \\ A^\top X+C \end{pmatrix}^\top P \begin{pmatrix} I \\ A^\top X+C \end{pmatrix} \prec 0$$

has a solution X. This implies that (4.4.17) has a solution for $B = I$. Therefore, with the Projection Lemma, (4.4.19) implies (4.4.20) for a general A.

In summary, we have proved for general A and B that (4.4.18) or (4.4.19) imply (4.4.20). We can infer the solvability of (4.4.17) or that of (7.2.10).   ■

4.4.4   Elimination for quadratic performance designs

In Section 4.2 we have derived that for the performance index

$$P = \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix}, \quad R \succcurlyeq 0 \quad \text{with inverse} \quad P^{-1} = \begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S'} & \tilde{R} \end{pmatrix}, \quad \tilde{Q} \preccurlyeq 0, \tag{4.4.21}$$

internal stability and quadratic performance can be achieved for the system (4.2.1) by a suitable controller (4.2.2) if and only if the synthesis inequalities

$$\left. \begin{matrix} \boldsymbol{X}(v) \succ 0 \\[2mm] \begin{pmatrix} I & 0 \\ \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ 0 & I \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{pmatrix}^\top \begin{pmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & Q & S \\ 0 & 0 & S^\top & R \end{pmatrix} \begin{pmatrix} I & 0 \\ \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ 0 & I \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{pmatrix} \prec 0 \end{matrix} \right\} \tag{4.4.22}$$

are feasible. Due to the specific structure

$$\left(\begin{array}{c|c} \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \hline \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{array} \right) = \left(\begin{array}{cc|c} AY & A & B \\ 0 & XA & XB_1 \\ \hline C_1Y & C_1 & D_1 \end{array} \right) + \begin{pmatrix} 0 & B \\ I & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} K & L \\ M & N \end{pmatrix} \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & C & F \end{array} \right),$$

it is straightforward to apply the Elimination Lemma 4.17 to eliminate all the variables K, L, M, N. For that purpose it suffices to compute basis matrices

$$\Phi = \begin{pmatrix} \Phi^1 \\ \Phi^2 \end{pmatrix} \text{ of } \ker \begin{pmatrix} B^\top & E^\top \end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix} \text{ of } \ker \begin{pmatrix} C & F \end{pmatrix}.$$

Corollary 4.18 *For a performance index with* (4.4.21)*, there exists a solution v of* (4.4.22) *if and only if there exist symmetric X and Y that satisfy*

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0, \tag{4.4.23}$$

$$\Psi^\top \begin{pmatrix} I & 0 \\ A & B_1 \\ \hline 0 & I \\ C_1 & D_1 \end{pmatrix}^\top \left(\begin{array}{cc|cc} 0 & X & 0 & 0 \\ X & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S \\ 0 & 0 & S^\top & R \end{array} \right) \begin{pmatrix} I & 0 \\ A & B_1 \\ \hline 0 & I \\ C_1 & D_1 \end{pmatrix} \Psi \prec 0, \tag{4.4.24}$$

$$\Phi^\top \begin{pmatrix} -A^\top & -C_1^\top \\ I & 0 \\ -B_1^\top & -D_1^\top \\ 0 & I \end{pmatrix}^\top \left(\begin{array}{cc|cc} 0 & Y & 0 & 0 \\ Y & 0 & 0 & 0 \\ \hline 0 & 0 & \check{Q} & \check{S} \\ 0 & 0 & \check{S}^\top & \check{R} \end{array} \right) \begin{pmatrix} -A^\top & -C_1^\top \\ I & 0 \\ -B_1^\top & -D_1^\top \\ 0 & I \end{pmatrix} \Phi \succ 0. \tag{4.4.25}$$

Remark 4.19 Note that the columns of $\begin{pmatrix} B \\ E \end{pmatrix}$ indicate in how far the right-hand side of (4.2.1) can be modified by control, and the rows of $\begin{pmatrix} C & F \end{pmatrix}$ determine those functionals that provide information about the system state and the disturbance that is available for control. Roughly speaking, the columns of Φ or of Ψ indicate what *cannot* be influenced by control or which information *cannot* be extracted from the measured output. Let us hence compare (4.4.23)-(4.4.25) with the synthesis inequalities that would be obtained for

$$\begin{pmatrix} \dot{x} \\ \hline z_1 \\ \vdots \\ z_q \end{pmatrix} = \left(\begin{array}{c|ccc} A & B_1 & \cdots & B_q \\ \hline C_1 & D_1 & \cdots & D_{1q} \\ \vdots & \vdots & \ddots & \vdots \\ C_q & D_{q1} & \cdots & D_q \end{array} \right) \begin{pmatrix} x \\ \hline w_1 \\ \vdots \\ w_q \end{pmatrix} \tag{4.4.26}$$

without control input and measurement output. For this system we could choose $\Phi = I$ and $\Psi = I$ to arrive at the synthesis inequalities

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0, \tag{4.4.27}$$

$$\begin{pmatrix} I & 0 \\ A & B_j \\ \hline 0 & I \\ C_1 & D_1 \end{pmatrix}^\top \left(\begin{array}{cc|cc} 0 & X & 0 & 0 \\ X & 0 & 0 & 0 \\ \hline 0 & 0 & Q_j & S_j \\ 0 & 0 & S_j^\top & R_j \end{array} \right) \begin{pmatrix} I & 0 \\ A & B_j \\ \hline 0 & I \\ C_1 & D_1 \end{pmatrix} \prec 0, \tag{4.4.28}$$

$$\begin{pmatrix} -A^\top & -C_1^\top \\ I & 0 \\ -B_j^\top & -D_1^\top \\ 0 & I \end{pmatrix}^\top \left(\begin{array}{cc|cc} 0 & Y & 0 & 0 \\ Y & 0 & 0 & 0 \\ \hline 0 & 0 & \check{Q}_j & \check{S}_j \\ 0 & 0 & \check{S}_j^\top & \check{R}_j \end{array} \right) \begin{pmatrix} -A^\top & -C_1^\top \\ I & 0 \\ -B_j^\top & -D_1^\top \\ 0 & I \end{pmatrix} \succ 0. \tag{4.4.29}$$

Since there is no control and no measured output, these could be viewed as analysis inequalities for (4.4.26). Hence we have very nicely displayed in how far controls or measurements do influence the synthesis inequalities through Φ and Ψ. Finally, we note that (4.4.27)-(4.4.29) are equivalent to $X \succ 0$, (4.4.28) or to $Y \succ 0$, (4.4.29). Moreover, if dualizing $X \succ 0$, (4.4.28), we arrive at $Y \succ 0$, (4.4.29) for $Y := X^{-1}$.

Proof of Corollary 4.18. The first inequality (4.4.23) is just $\boldsymbol{X}(v) \succ 0$. The inequalities (4.4.24)-(4.4.25) are obtained by simply applying Lemma 4.17 to the second inequality of (4.4.22), viewed as a quadratic matrix inequality in the unknowns K, L, M, N. For that purpose we first observe that

$$\ker \begin{pmatrix} 0 & I & \bigg| & 0 \\ B^\top & 0 & \bigg| & E^\top \end{pmatrix} \quad \text{has the basis matrix} \quad \begin{pmatrix} \Phi^1 \\ 0 \\ \hline \Phi^2 \end{pmatrix}$$

$$\ker \begin{pmatrix} I & 0 & \bigg| & 0 \\ 0 & C & \bigg| & F \end{pmatrix} \quad \text{has the basis matrix} \quad \begin{pmatrix} 0 \\ \Psi^1 \\ \hline \Psi^2 \end{pmatrix}$$

Since

$$\begin{pmatrix} I & 0 \\ \hline \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \hline 0 & I \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{pmatrix} \begin{pmatrix} 0 \\ \Psi^1 \\ \hline \Psi^2 \end{pmatrix} = \begin{pmatrix} I & 0 & \bigg| & 0 \\ 0 & I & \bigg| & 0 \\ AY & A & \bigg| & B_1 \\ 0 & XA & \bigg| & XB_1 \\ \hline 0 & 0 & \bigg| & I \\ C_1Y & C_1 & \bigg| & D_1 \end{pmatrix} \begin{pmatrix} 0 \\ \Psi^1 \\ \hline \Psi^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ I & 0 \\ A & B_1 \\ XA & XB_1 \\ \hline 0 & I \\ C_1 & D_1 \end{pmatrix} \Psi,$$

the solvability condition that corresponds to the first inequality in (4.4.16) reads as

$$\Psi^\top \begin{pmatrix} 0 & 0 \\ I & 0 \\ A & B_1 \\ XA & XB_1 \\ \hline 0 & I \\ C_1 & D_1 \end{pmatrix}^\top \begin{pmatrix} 0 & 0 & I & 0 & \bigg| & 0 & 0 \\ 0 & 0 & 0 & I & \bigg| & 0 & 0 \\ I & 0 & 0 & 0 & \bigg| & 0 & 0 \\ 0 & I & 0 & 0 & \bigg| & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \bigg| & Q & S \\ 0 & 0 & 0 & 0 & \bigg| & S^\top & R \end{pmatrix} \begin{pmatrix} 0 & 0 \\ I & 0 \\ A & B_1 \\ XA & XB_1 \\ \hline 0 & I \\ C_1 & D_1 \end{pmatrix} \Psi \prec 0.$$

This simplifies to

$$\Psi^\top \begin{pmatrix} I & 0 \\ XA & XB_1 \\ \hline 0 & I \\ C_1 & D_1 \end{pmatrix}^\top \begin{pmatrix} 0 & I & \bigg| & 0 & 0 \\ I & 0 & \bigg| & 0 & 0 \\ \hline 0 & 0 & \bigg| & Q & S \\ 0 & 0 & \bigg| & S^\top & R \end{pmatrix} \begin{pmatrix} I & 0 \\ XA & XB_1 \\ \hline 0 & I \\ C_1 & D_1 \end{pmatrix} \Psi \prec 0.$$

This is clearly nothing but (4.4.24). The very same steps lead to (4.4.25). Indeed, we have

$$\begin{pmatrix} \boldsymbol{A}(v)^\top & \boldsymbol{C}(v)^\top \\ I & 0 \\ \hline \boldsymbol{B}(v)^\top & \boldsymbol{D}(v)^\top \\ 0 & I \end{pmatrix} \begin{pmatrix} \Phi^1 \\ 0 \\ \hline \Phi^2 \end{pmatrix} =$$

$$= \begin{pmatrix} -YA^\top & 0 & \bigg| & -YC_1^\top \\ -A^\top & -A^\top X & \bigg| & -C_1^\top \\ I & 0 & \bigg| & 0 \\ 0 & I & \bigg| & 0 \\ \hline -B_1^\top & -XB_1^\top & \bigg| & -D_1^\top \\ 0 & 0 & \bigg| & I \end{pmatrix} \begin{pmatrix} \Phi^1 \\ 0 \\ \hline \Phi^2 \end{pmatrix} = \begin{pmatrix} -YA^\top & -YC_1^\top \\ -A^\top & -C_1^\top \\ I & 0 \\ 0 & 0 \\ \hline -B_1^\top & -D_1^\top \\ 0 & I \end{pmatrix} \Phi$$

such that the solvability condition that corresponds to the second inequality in (4.4.16) is

$$
\Phi^\top \begin{pmatrix} -YA^\top & -YC_1^\top \\ -A^\top & -C_1^\top \\ I & 0 \\ 0 & 0 \\ \hline -B_1^\top & -D_1^\top \\ 0 & I \end{pmatrix}^\top \left(\begin{array}{cccc|cc} 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \tilde{Q} & \tilde{S} \\ 0 & 0 & 0 & 0 & \tilde{S}^\top & \tilde{R} \end{array}\right) \begin{pmatrix} -YA^\top & -YC_1^\top \\ -A^\top & -C_1^\top \\ I & 0 \\ 0 & 0 \\ \hline -B_1^\top & -D_1^\top \\ 0 & I \end{pmatrix} \Phi \prec 0
$$

which simplifies to

$$
\Phi^\top \begin{pmatrix} -YA^\top & -YC_1^\top \\ I & 0 \\ \hline -B_1^\top & -D_1^\top \\ 0 & I \end{pmatrix}^\top \left(\begin{array}{cc|cc} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{Q} & \tilde{S} \\ 0 & 0 & \tilde{S}^\top & \tilde{R} \end{array}\right) \begin{pmatrix} -YA^\top & -YC_1^\top \\ I & 0 \\ \hline -B_1^\top & -D_1^\top \\ 0 & I \end{pmatrix} \Phi \prec 0
$$

and we arrive at (4.4.25).   ∎

Starting from the synthesis inequalities (4.4.22) in the variables X, Y, K, L, M, N, we have derived the equivalent inequalities (4.4.23)-(4.4.25) in the variables X, Y only. Testing feasibility of these latter inequalities can hence be accomplished much faster. This is particularly advantageous if optimizing an additional parameter, such as minimizing the sup-optimality level γ in the H_∞ problem.

To conclude this section, let us comment on how to compute the controller after having found solutions X, Y of (4.4.23)-(4.4.25). One possibility is to explicitly solve the quadratic inequality (4.4.22) in K, L, M, N along the lines of the proof of Lemma 4.17, and reconstruct the controller parameters as earlier. One could as well proceed directly: Starting from X and Y, we can compute non-singular U and V with $UV^\top = I - XY$, and determine $\mathscr{X} \succ 0$ by solving the first equation in (4.2.14). Due to (4.2.4), we can apply Lemma 4.17 directly to the analysis inequality

$$
\begin{pmatrix} I & 0 \\ \mathscr{A} & \mathscr{B} \\ \hline 0 & I \\ \mathscr{C} & \mathscr{D} \end{pmatrix}^\top \left(\begin{array}{cc|cc} 0 & \mathscr{X} & 0 & 0 \\ \mathscr{X} & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S \\ 0 & 0 & S^\top & R \end{array}\right) \begin{pmatrix} I & 0 \\ \mathscr{A} & \mathscr{B} \\ \hline 0 & I \\ \mathscr{C} & \mathscr{D} \end{pmatrix} \prec 0
$$

if viewing A_c, B_c, C_c, D_c as variables. It is not difficult (and you are invited to provide the details!) to verify the solvability conditions for this quadratic inequality, and to construct an explicit solution along the lines of the proof of Lemma 4.17. Alternatively, one can transform the quadratic inequality to a linear matrix inequality with Lemma 4.1, and apply the Projection Lemma to reconstruct the controller parameters. We conclude that there are many basically equivalent alternative ways to compute a controller once one has determined X and Y.

4.4.5   Elimination for H_2 designs

If recalling (4.2.8), we observe that the synthesis inequalities (4.2.19) in the H_2 controller design problems involve the variables M and N, but only the first inequality

$$\begin{pmatrix} \boldsymbol{A}(v)^\top + \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{B}(v)^\top & -\gamma I \end{pmatrix} \prec 0 \tag{4.4.30}$$

is affected by K and L. This might suggest that the latter two variables can be eliminated in the synthesis conditions. Since (4.4.30) is affine in $\begin{pmatrix} K & L \end{pmatrix}$, we can indeed apply the Projection Lemma to eliminate these variables. It is not difficult to arrive at the following alternative synthesis conditions for H_2-type criteria.

Corollary 4.20 *There exists a controller that renders* (4.2.18) *for some* $\mathscr{X}$, Z *satisfied if and only if there exist* X, Y, M, N, Z *with* $f(Z) < \gamma$, $D_1 + ENF = 0$ *and*

$$\begin{pmatrix} Y & I & (C_1 Y + EM)^\top \\ I & X & (C_1 + ENC)^\top \\ C_1 Y + EM & C_1 + ENC & Z \end{pmatrix} \succ 0,$$

$$\Psi^\top \begin{pmatrix} A^\top X + XA & XB_1 \\ B_1^\top X & -\gamma I \end{pmatrix} \Psi \prec 0, \quad \begin{pmatrix} (AY + BM) + (AY + BM)^\top & B_1 + BNF \\ (B_1 + BNF)^\top & -\gamma I \end{pmatrix} \prec 0. \tag{4.4.31}$$

Proof. We only need to show that the elimination of K and L in (4.4.30) leads to the two inequalities (4.4.31). Let us recall

$$\begin{pmatrix} \boldsymbol{A}(v) & \boldsymbol{B}(v) \end{pmatrix} = \left(\begin{array}{cc|c} AY & A & B_1 \\ 0 & XA & XB_1 \end{array} \right) + \begin{pmatrix} 0 & B \\ I & 0 \end{pmatrix} \begin{pmatrix} K & L \\ M & N \end{pmatrix} \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & C & F \end{array} \right) =$$

$$= \left(\begin{array}{cc|c} AY + BM & A + BNC & B_1 + BNF \\ 0 & XA & XB_1 \end{array} \right) + \begin{pmatrix} 0 \\ I \end{pmatrix} \begin{pmatrix} K & L \end{pmatrix} \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & C & F \end{array} \right).$$

Therefore, (4.4.30) is equivalent to

$$\left(\begin{array}{cc|c} AY + YA^\top & A & B_1 \\ A^\top & A^\top X + XA & XB_1 \\ \hline B_1^\top & B_1^\top X & -\gamma I \end{array} \right) + \mathrm{sym}\left(\begin{pmatrix} B \\ 0 \\ \hline 0 \end{pmatrix} \begin{pmatrix} M & N \end{pmatrix} \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & C & F \end{array} \right) \right) +$$

$$+ \mathrm{sym}\left(\begin{pmatrix} 0 \\ I \\ \hline 0 \end{pmatrix} \begin{pmatrix} K & L \end{pmatrix} \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & C & F \end{array} \right) \right) \prec 0$$

where $\mathrm{sym}(M) := M + M^\top$ is just an abbreviation to shorten the formulas.

Now note that

$$
\ker \begin{pmatrix} 0 & I & | & 0 \end{pmatrix} \quad \text{has the basis matrix} \quad \begin{pmatrix} I & 0 \\ 0 & 0 \\ \hline 0 & I \end{pmatrix}
$$

$$
\ker \begin{pmatrix} I & 0 & | & 0 \\ 0 & C & | & F \end{pmatrix} \quad \text{has the basis matrix} \quad \begin{pmatrix} 0 \\ \Psi^1 \\ \hline \Psi^2 \end{pmatrix}
$$

Therefore, the Projection Lemma leads to the two inequalities

$$
\begin{pmatrix} 0 \\ \Psi^1 \\ \hline \Psi^2 \end{pmatrix}^\top
\begin{pmatrix} AY + YA^\top & A & | & B_1 \\ A^\top & A^\top X + XA & | & XB_1 \\ \hline B_1^\top & B_1^\top X & | & -\gamma I \end{pmatrix}
\begin{pmatrix} 0 \\ \Psi^1 \\ \hline \Psi^2 \end{pmatrix} \prec 0
$$

and

$$
\begin{pmatrix} AY + YA^\top & B_1 \\ B_1^\top & -\gamma I \end{pmatrix} + \mathrm{sym}\left(\begin{pmatrix} B \\ 0 \end{pmatrix} \begin{pmatrix} M & N \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & F \end{pmatrix} \right) \prec 0
$$

that are easily rewritten to (4.4.31). ∎

If $E = 0$, we can actually eliminate all variables K, L, M, N from the synthesis inequalities. The corresponding results are obtained in a straightforward fashion and their proof is left as an exercise.

Corollary 4.21 *Suppose that $E = 0$. Then there exists a controller that renders (4.2.18) for some $\mathcal{X}$, Z satisfied if and only if $D_1 = 0$ and there exist X, Y Z with $f(Z) < \gamma$ and*

$$
\begin{pmatrix} Y & I & (C_1 Y)^\top \\ I & X & C_1^\top \\ C_1 Y & C_1 & Z \end{pmatrix} \succ 0,
$$

$$
\Psi^\top \begin{pmatrix} A^\top X + XA & XB_1 \\ B_1^\top X & -\gamma I \end{pmatrix} \Psi \prec 0, \quad \begin{pmatrix} \widehat{\Phi} & 0 \\ 0 & I \end{pmatrix}^\top \begin{pmatrix} AY + YA^\top & B_1 \\ B_1^\top & -\gamma I \end{pmatrix} \begin{pmatrix} \widehat{\Phi} & 0 \\ 0 & I \end{pmatrix} \prec 0
$$

where $\widehat{\Phi}$ is a basis matrix of $\ker(B)$.

Remark 4.22 Once the synthesis inequalities have been solved, the computation of $\begin{pmatrix} K & L \end{pmatrix}$ or of the parameters K, L, M, N can be performed along the lines that are indicated in the proof of the Projection Lemma.

Remark 4.23 In this section we have focused on performing variable elimination which involve as little computations as possible. These results should be viewed as examples on how one can proceed in specific circumstances. The examples can be easily extended to various other performance specifications. As an exercise, the reader is invited to eliminate decision variables in the synthesis results that we derived for controller designs that bound the peak-to-peak norm of the controlled system.

4.5 State-feedback problems

The state-feedback problem is characterized by

$$y = x \quad \text{or} \quad \begin{pmatrix} C & F_1 & \cdots & F_q \end{pmatrix} = \begin{pmatrix} I & 0 & \cdots & 0 \end{pmatrix}.$$

Then the formulas (4.2.8) read as

$$\begin{pmatrix} \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{pmatrix} = \left(\begin{array}{cc|c} AY + BM & A + BN & B_1 \\ K & AX + L & XB_1 \\ \hline C_1 Y + EM & C_1 + EN & D_1 \end{array} \right).$$

Note that the variable L only appears in the $(2,2)$-block, and that we can assign an arbitrary matrix in this position by suitably choosing L. Therefore, by varying L, the $(2,2)$ block of

$$\begin{pmatrix} \boldsymbol{A}(v)^\top + \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{B}(v) & 0 \end{pmatrix} =$$

$$= \left(\begin{array}{cc|c} (AY + BM) + (AY + BM)^\top & (A + BN) + K^\top & B_1 \\ K + (A + BN)^\top & (AX + L) + (AX + L)^\top & XB_1 \\ \hline B_1^\top & B_1^\top X & 0 \end{array} \right)$$

varies in the set of all symmetric matrices. This allows to apply Lemma 4.13 in order to eliminate L in synthesis inequalities and leads to a drastic simplification.

Let us illustrate all this for the quadratic performance problem. The corresponding synthesis inequalities (4.2.12) read as

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0, \quad \left(\begin{array}{cc|c} (AY + BM) + (AY + BM)^\top & (A + BN) + K^\top & B_1 \\ K + (A + BN)^\top & (AX + L) + (AX + L)^\top & XB_1 \\ \hline B_1^\top & B_1^\top X & 0 \end{array} \right) +$$

$$+ \left(\begin{array}{cc|c} 0 & 0 & I \\ C_1 Y + EM & C_1 + EN & D_1 \end{array} \right)^\top P \left(\begin{array}{cc|c} 0 & 0 & I \\ C_1 Y + EM & C_1 + EN & D_1 \end{array} \right) \prec 0.$$

These imply, just by canceling the second block row/column,

$$Y \succ 0, \quad \left(\begin{array}{c|c} (AY + BM) + (AY + BM)^\top & B_1 \\ \hline B_1^\top & 0 \end{array} \right) +$$

$$+ \left(\begin{array}{c|c} 0 & I \\ C_1 Y + EM & D_1 \end{array} \right)^\top P \left(\begin{array}{c|c} 0 & I \\ C_1 Y + EM & D_1 \end{array} \right) \prec 0$$

or

$$Y \succ 0, \quad \left(\begin{array}{cc} I & 0 \\ AY + BM & B_1 \\ 0 & I \\ C_1 Y + EM & D_1 \end{array} \right)^\top \left(\begin{array}{cc|cc} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S \\ 0 & 0 & S^\top & R \end{array} \right) \left(\begin{array}{cc} I & 0 \\ AY + BM & B_1 \\ 0 & I \\ C_1 Y + EM & D_1 \end{array} \right) \prec 0. \quad (4.5.1)$$

This is a drastic simplification since only the variables Y and M do appear in the resulting inequalities. It is no problem to reverse the arguments in order to show that the reduced inequalities are equivalent to the full synthesis inequalities.

However, proceeding in a different fashion leads to another fundamental insight: With solutions Y and M of (4.5.1), one can in fact design a *static* controller which solves the quadratic performance problem. Indeed, we just choose

$$D_c := MY^{-1}$$

to infer that the static controller $y = D_c u$ leads to a controlled system with the describing matrices

$$\begin{pmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{pmatrix} = \begin{pmatrix} A + BD_c & B_1 \\ C_1 + ED_c & D_1 \end{pmatrix} = \begin{pmatrix} (AY + BM)Y^{-1} & B_1 \\ (C_1 Y + EM)Y^{-1} & D_1 \end{pmatrix}.$$

We infer that (4.5.1) is identical to

$$Y \succ 0, \quad \begin{pmatrix} I & 0 \\ \mathscr{A}Y & \mathscr{B} \\ \hline 0 & I \\ \mathscr{C}Y & \mathscr{D} \end{pmatrix}^{\top} \begin{pmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S \\ 0 & 0 & S^{\top} & R \end{pmatrix} \begin{pmatrix} I & 0 \\ \mathscr{A}Y & \mathscr{B} \\ \hline 0 & I \\ \mathscr{C}Y & \mathscr{D} \end{pmatrix} \prec 0.$$

If we perform congruence transformations with Y^{-1} and $\begin{pmatrix} Y^{-1} & 0 \\ 0 & I \end{pmatrix}$, we arrive with $\mathscr{X} := Y^{-1}$ at

$$\mathscr{X} \succ 0, \quad \begin{pmatrix} I & 0 \\ \mathscr{X}\mathscr{A} & \mathscr{X}\mathscr{B} \\ \hline 0 & I \\ \mathscr{C} & \mathscr{D} \end{pmatrix}^{\top} \begin{pmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S \\ 0 & 0 & S^{\top} & R \end{pmatrix} \begin{pmatrix} I & 0 \\ \mathscr{X}\mathscr{A} & \mathscr{X}\mathscr{B} \\ \hline 0 & I \\ \mathscr{C} & \mathscr{D} \end{pmatrix} \prec 0.$$

Hence the static gain $\mathscr{D}$ indeed defines a controller which solves the quadratic performance problem.

Corollary 4.24 *Under the state-feedback information structure, there exists a dynamic controller* (4.2.2) *and some $\mathscr{X}$ which satisfy* (4.2.6) *if and only if there exist solutions Y and M of the inequalities* (4.5.1). *If Y and M solve* (4.5.1), *the* static *state-feedback controller gain*

$$D_c = MY^{-1}$$

and the Lyapunov matrix $\mathscr{X} := Y^{-1}$ render (4.2.6) *satisfied.*

In literally the same fashion as for output-feedback control, we arrive at the following general procedure to proceed from analysis inequalities to synthesis inequalities, and to construct a static state-feedback controller:

Algorithm 4.25 (State feedback synthesis algorithm) Aim: Synthesis of general static state feedback control laws.

Step 1: Rewrite the analysis inequalities in the blocks $\mathscr{X}$, $\mathscr{X}\mathscr{A}$, $\mathscr{X}\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$ in order to be able to find a (formal) congruence transformation involving $\mathscr{Y}$ which leads to inequalities in the blocks $\mathscr{Y}^\top \mathscr{X}\mathscr{Y}$, $\mathscr{Y}^\top \mathscr{X}\mathscr{A}\mathscr{Y}$, $\mathscr{Y}^\top \mathscr{X}\mathscr{B}$, $\mathscr{C}\mathscr{Y}$, $\mathscr{D}$.

Step 2: Perform the substitutions

$$\mathscr{Y}^\top \mathscr{X}\mathscr{Y} \to Y \ \text{ and } \ \begin{pmatrix} \mathscr{Y}^\top \mathscr{X}\mathscr{A}\mathscr{Y} & \mathscr{Y}^\top \mathscr{X}\mathscr{B} \\ \mathscr{C}\mathscr{Y} & \mathscr{D} \end{pmatrix} \to \begin{pmatrix} AY + BM & B_1 \\ C_1 Y + EM & D_1 \end{pmatrix}$$

to arrive at matrix inequalities in the variables Y and M.

Step 3: Solve the synthesis inequalities for Y and M.

Step 4: Set the static controller gain $\mathscr{D}$ and the Lyapunov matrix $\mathscr{X}$ to

$$\mathscr{D} = MY^{-1} \ \text{ and } \ \mathscr{X} = Y^{-1}.$$

These matrices render the analysis inequalities satisfied.

As an illustration, starting form the analysis inequalities (4.2.18) for H_2-type synthesis problems, the corresponding state-feedback synthesis conditions read as

$$\begin{pmatrix} (AY + BM)^\top + (AY + BM) & B_1 \\ B_1^\top & -\gamma I \end{pmatrix} \prec 0,$$

$$\begin{pmatrix} Y & (C_1 Y + EM)^\top \\ C_1 Y + EM & Z \end{pmatrix} \succ 0, \ f(Z) < \gamma, \ D_1 = 0.$$

All our previous remarks pertaining to the (more complicated) procedure for the output-feedback information structure apply without modification.

In general we can conclude that dynamics in the controller do not offer any advantage over static controllers for state-feedback problems. This is also true for mixed control problems. This statements requires extra attention since our derivation was based on eliminating the variable L which might occur in several matrix inequalities. At this point the remark after Lemma 4.13 comes into play: This particular elimination result also applies to systems of matrix inequalities such that, indeed, the occurrence of L is various inequalities will not harm the arguments.

As earlier, in the single-objective quadratic performance problem by state-feedback, it is possible to eliminate the variable M in (4.5.1). Alternatively, one could as well exploit the particular structure of the system description to simplify the conditions in Theorem 4.18. Both approaches lead to the following result.

Corollary 4.26 *For the state-feedback quadratic performance problem with index satisfying* (4.4.21)*, there exists dynamic controller and some $\mathscr{X}$ with* (4.2.6) *if and only if there exists a symmetric Y which solves*

$$Y \succ 0, \ \Phi^\top \begin{pmatrix} -A^\top & -C_1^\top \\ I & 0 \\ -B_1^\top & -D_1^\top \\ 0 & I \end{pmatrix}^\top \left(\begin{array}{cc|cc} 0 & Y & 0 & 0 \\ Y & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{Q} & \tilde{S} \\ 0 & 0 & \tilde{S}^\top & \tilde{R} \end{array} \right) \begin{pmatrix} -A^\top & -C_1^\top \\ I & 0 \\ -B_1^\top & -D_1^\top \\ 0 & I \end{pmatrix} \Phi \succ 0. \qquad (4.5.2)$$

Remark 4.27 The state feedback results in this subsection should be viewed as illustrations on how to proceed for specific system descriptions. Indeed, another popular choice is the so-called *full information* structure in which both the state and the disturbance are measurable:

$$y = \begin{pmatrix} x \\ w \end{pmatrix}.$$

Similarly, one could consider the corresponding dual versions that are typically related to *estimation problems*, such as e.g.

$$\begin{pmatrix} B \\ E_1 \\ \vdots \\ E_q \end{pmatrix} = \begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We have collected all auxiliary results that allow to handle these specific problems without any complications.

4.6 Discrete-time systems

Everything that has been said so far can be easily extended to discrete time-design problems. This is particularly surprising since, in the literature, discrete-time problem solutions often seem much more involved and harder to master than their continuous-time counterparts.

Our general procedure to step from analysis to synthesis as well as the technique to recover the controller need no change at all; in particular, the concrete formulas for the block substitutions do not change. The elimination of transformed controller parameters proceeds in the same fashion on the basis of the Projection Lemma or the Elimination Lemma and the specialized version thereof.

Only as an example we consider the problem discussed in [21]: the mixed H_2/H_∞ problem with different disturbance inputs and controlled outputs in discrete-time.

It is well-known [21] that $\mathscr{A}$ has all its eigenvalues in the unit disk, that the discrete time H_2-norm of

$$\mathscr{C}_1(zI - \mathscr{A})^{-1}\mathscr{B}_1 + \mathscr{D}_1$$

is smaller than γ_1, and that the discrete time H_∞-norm of

$$\mathscr{C}_2(zI - \mathscr{A})^{-1}\mathscr{B}_2 + \mathscr{D}_2$$

is smaller than γ_2 if and only if there exist symmetric matrices $\mathscr{X}_1$, $\mathscr{X}_2$, and Z with $\mathrm{trace}(Z) < \gamma_1$ and

$$\begin{pmatrix} \mathscr{X}_1 & \mathscr{X}_1\mathscr{A} & \mathscr{X}_1\mathscr{B}_1 \\ \mathscr{A}^\top\mathscr{X}_1 & \mathscr{X}_1 & 0 \\ \mathscr{B}_1^\top\mathscr{X}_1 & 0 & \gamma_1 I \end{pmatrix} \succ 0, \quad \begin{pmatrix} \mathscr{X}_1 & 0 & \mathscr{C}_1^\top \\ 0 & I & \mathscr{D}_1^\top \\ \mathscr{C}_1 & \mathscr{D}_1 & Z \end{pmatrix} \succ 0, \quad \begin{pmatrix} \mathscr{X}_2 & 0 & \mathscr{A}^\top\mathscr{X}_2 & \mathscr{C}_2^\top \\ 0 & \gamma_2 I & \mathscr{B}_2^\top\mathscr{X}_2 & \mathscr{D}_2^\top \\ \mathscr{X}_2\mathscr{A} & \mathscr{X}_2\mathscr{B}_2 & \mathscr{X}_2 & 0 \\ \mathscr{C}_2 & \mathscr{D}_2 & 0 & \gamma_2 I \end{pmatrix} \succ 0.$$

$$(4.6.1)$$

Note that we have transformed these analysis LMI's such that they are affine in the blocks that will be transformed for synthesis.

The mixed problem consists of searching for a controller that renders these inequalities satisfied with a common Lyapunov function $\mathscr{X} = \mathscr{X}_1 = \mathscr{X}_2$. The solution is immediate: Perform congruence transformations of (4.6.1) with

$$\text{diag}(\mathscr{Y},\mathscr{Y},I), \;\; \text{diag}(\mathscr{Y},I,I), \;\; \text{diag}(\mathscr{Y},I,\mathscr{Y},I)$$

and read off the synthesis LMI's using (4.2.8). After solving the synthesis LMI's, we stress again that the controller construction proceeds along *the same steps* as in Theorem 4.2. The inclusion of pole constraints for arbitrary LMI regions (related, of course, to discrete time stability) and other criteria poses no extra problems.

4.7   Further reading

The linearization and convexification transformation that is at the basis of the synthesis algorithms discussed in this Chapter have been published in its general format in [28, 50]. Details on the Youla parametrization can be found in [26]. For more information on multi-objective and mixed control synthesis problems, the reader is referred to [16, 43, 44, 53]. Mixed control problems for discrete time systems are discussed in [21]. for more details on the dualization of matrix inequalities we refer to [18, 45]. We refer to [40] for an early reference to generalized H_2 optimization. The Elimination Lemma 4.17 has been published in [49].

4.8   Exercises

Exercise 1
Derive an LMI solution of the H_∞-control problem for the system

$$\begin{pmatrix} \dot{x} \\ z_1 \\ y \end{pmatrix} = \begin{pmatrix} A & B_1 & B \\ C_1 & D_1 & E \\ C & F & 0 \end{pmatrix} \begin{pmatrix} x \\ w_1 \\ u \end{pmatrix}$$

with

$$C = \begin{pmatrix} I \\ 0 \end{pmatrix}, \;\; F = \begin{pmatrix} 0 \\ I \end{pmatrix} \;\; \text{such that} \;\; y = \begin{pmatrix} x \\ w_1 \end{pmatrix}.$$

(This is the so-called full information problem.)

Exercise 2 (H_∞ estimator design)
Consider the system

$$\begin{pmatrix} \dot{x} \\ z \\ y \end{pmatrix} = \begin{pmatrix} A & B_1 \\ C_1 & D_1 \\ C & F \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}$$

and interconnect it with the estimator

$$\begin{pmatrix} \dot{x}_c \\ \hat{z} \end{pmatrix} = \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} \begin{pmatrix} x_c \\ y \end{pmatrix} \tag{4.8.1}$$

where both A and A_c are Hurwitz. The goal in optimal estimation is to design an estimator which keeps $z - \hat{z}$ as small as possible for all disturbances w in a certain class. Out of the multitude of possibilities, we choose the $\mathcal{L}_2$-gain of $w \to z - \hat{z}$ (for zero initial condition of both the system and the estimator) as the measure of the estimation quality.

This leads to the following problem formulation: Given $\gamma > 0$, test whether there exists an estimator which renders

$$\sup_{w \in \mathcal{L}_2, \, w \neq 0} \frac{\|z - \hat{z}\|_2}{\|w\|_2} < \gamma \tag{4.8.2}$$

satisfied. If yes, reveal how to design an estimator that leads to this property.

(a) Show that the estimation problem is a specialization of the general output-feedback H_∞-design problem.

(b) Due to the specific structure of the open-loop system, show that there exists a linearizing transformation of the estimator parameters which does not involve any matrices that describe the open-loop system.

Hint: To find the transformation, proceed as in the proof of Theorem 4.2 with the factorization

$$\mathscr{Y}^\top \mathscr{X} = \mathscr{Z} \text{ where } \mathscr{Y}^\top = \begin{pmatrix} I & Y^{-1}V \\ I & 0 \end{pmatrix}, \quad \mathscr{Z} = \begin{pmatrix} Y^{-1} & 0 \\ X & U \end{pmatrix},$$

Exercise 3 (LQ optimal control)
Suppose we are given the system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \quad x(0) = \xi$$

with a stabilizable pair (A, B). For a fixed initial condition ξ let us define the optimal LQ cost

$$J(\xi) := \inf_{F, \, A+BF \text{ is stable}} \int_0^\infty y(t)^\top y(t) \, dt.$$

(a) Let y_F denote the system output for a fixed F for which $A + BF$ is stable. Show that

$$(A + BF)^\top K + K(A + BF) + (C + DF)^\top (C + DF) = 0$$

has a unique solution $K_F \succcurlyeq 0$ and that

$$\int_0^\infty y_F(t)^\top y_F(t) \, dt = \xi^\top K_F \xi.$$

(b) Let $y_F(.)$ denote the system output for a fixed F such that $A+BF$ is stable. Show that

$$\int_0^\infty y_F(t)^\top y_F(t)\,dt = \inf_{K \succ 0 \text{ satisfies } (A+BF)^\top K + K(A+BF) + (C+DF)^\top(C+DF) \prec 0} \xi^\top K \xi.$$

(c) Show that

$$J(\xi) = \inf_{F,\ K \succ 0 \text{ satisfies } (A+BF)^\top K + K(A+BF) + (C+DF)^\top(C+DF) \prec 0} \xi^\top K \xi.$$

(d) Derive an LMI problem to compute $J(\xi)$.

Exercise 4 (Time-delay systems)
For some $\tau > 0$ consider the time-delay system

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) + Gw(t) + Bu(t), \quad z(t) = Cx(t) + Du(t).$$

Given any continuous function ϕ on $[-\tau,0]$ and any continuous disturbance $w(.)$ on $[0,\infty)$, the initialization $x(t) = \phi(t)$ for $t \in [-\tau,0]$ leads to a unique solution of the differential equation. This is an infinite dimensional system with state-space equal to all continuous functions ξ on $[-\tau,0]$. For symmetric K and K_d the expression

$$\xi(0)^\top K \xi(0) + \int_{-\tau}^0 \xi(\sigma)^\top K_d \xi(\sigma)\,d\sigma$$

defines a mapping from this function space into the real numbers and is hence storage function candidate, called Lyapunov-Krasovski functional. The value at time t of this storage function along a system trajectory $x(.)$ is then given by

$$v(t) := x(t)^\top K x(t) + \int_{t-\tau}^t x(\sigma)^\top K_d x(\sigma)\,d\sigma.$$

(a) For $u(t) = 0$ and $w(t) = 0$ show that

$$\frac{d}{dt}v(t) = \begin{pmatrix} x(t) \\ x(t-\tau) \end{pmatrix}^\top \begin{pmatrix} A^\top K + KA + K_d & KA_d \\ A_d^\top K & -K_d \end{pmatrix} \begin{pmatrix} x(t) \\ x(t-\tau) \end{pmatrix}.$$

(b) Suppose that $u(t) = 0$ and $w(t) = 0$. Show that, if there exist $K \succ 0$ and $K_d \succ 0$ with

$$\begin{pmatrix} A^\top K + KA + K_d & KA_d \\ A_d^\top K & -K_d \end{pmatrix} \prec 0,$$

then all solutions satisfy $\lim_{t\to\infty} x(t) = 0$. To prove this, you may like to proceed along the following lines:

- Show that there exists an $\varepsilon > 0$ such that $\varepsilon \|x(t)\|^2 \leq v(t)$ and $\dot{v}(t) \leq -\varepsilon \|x(t)\|^2$.
- Show that $x(\cdot)$ and hence (via the differential equation) $\dot{x}(\cdot)$ are bounded on $[0,\infty)$.

- If $\lim_{t\to\infty} x(t) = 0$ is not true, there exist a constant $\alpha > 0$ and a sequence $t_k \to \infty$ such that $\|x(t_k)\| \geq \alpha$ for all k. Show that there exists some β such that $\|x(t)\| \geq \frac{1}{2}\alpha$ for all $t \in [t_k - \beta, t_k + \beta]$ and all k.

- Show that $\dot{v}(t) \leq -\varepsilon\alpha^2/4$ for all $t \in [t_k - \beta, t_k + \beta]$ and all k. Note that one can make sure that the intervals $[t_k - \beta, t_k + \beta]$ do not overlap for $k = 1, 2, \ldots$, by choosing a subsequence if necessary. Let's assume that this has been done.

- Show that $v(t_{k+1} - \beta) \leq -\beta\varepsilon\alpha^2/2 + v(t_k - \beta)$.

- Show that there exists a k with $v(t_k - \beta) < 0$. Why is this a contradiction?

Note that these are standard Lyapunov arguments (for delay systems) which actually hold in much larger generality!

(c) Suppose that $u(t) = 0$ for $t \geq 0$ and $x(t) = 0$ for $t \in [-\tau, 0]$. Assume that the disturbance $w(\cdot) \in \mathcal{L}_2$ is of finite energy. Show that, if there exist $K \succ 0$ and $K_d \succ 0$ with

$$\begin{pmatrix} A^\top K + KA + K_d & KA_d & KG \\ A_d^\top K & -K_d & 0 \\ G^\top K & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & I \\ C & 0 & D \end{pmatrix}^\top \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & 0 & I \\ C & 0 & D \end{pmatrix} \prec 0,$$

then the $\mathcal{L}_2$-gain of the system is not larger than one.

(d) Derive a sufficient LMI condition for the existence of a state-feedback controller which stabilizes the system and which guarantees that the $\mathcal{L}_2$-gain of the system is bounded by one.

Exercise 5 (H_2-observer synthesis)

In this exercise we consider the design of an optimal H_2-observer for the configuration
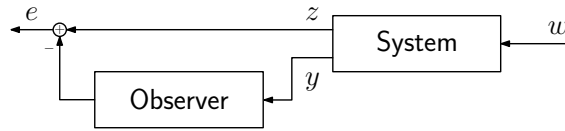
The system is described as

$$\begin{pmatrix} \dot{x} \\ z \\ y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \\ E & F \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}$$

where y is the measurement, z is the to-be-estimated variable, and w is white noise. The observer is assumed to admit the description

$$\dot{\xi} = A\xi + L(E\xi - y), \quad \hat{z} = C\xi$$

where the observer gain L is the design parameter.

(a) With the estimation error $e = z - \hat{z}$, show that the transfer matrix $w \to e$ admits the realization

$$\left[\begin{array}{c|c} A + LE & B + LF \\ \hline C & D \end{array}\right].$$

(b) For a fixed observer gain L formulate an LMI problem that allows to analyze whether $A + LE$ is Hurwitz and whether the H_2-norm of $w \to e$ is strictly smaller than some $\gamma > 0$.

(c) Formulate an LMI problem for computing an observer gain L such that $A + LE$ is Hurwitz and such that the H_2-norm of $w \to e$ is as small as possible.

(d) Consider the mechanical system
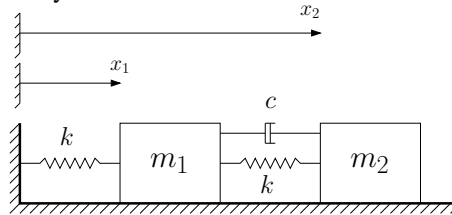
with the numerical values $m_1 = 1$, $m_2 = 0.5$, $k = 1$ and $c = 2$. The goal is to estimate the position of the second mass, on the basis of a measurement of the position of the first mass. Both the first mass and the position measurement are affected by white noise. With the given numerical values the system is modeled as

$$
\begin{pmatrix} \dot{x} \\ z \\ y \end{pmatrix} = \left(\begin{array}{cccc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & -2 & 2 & 1 & 0 \\ 2 & -2 & 4 & -4 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0.1 \end{array} \right) \begin{pmatrix} x \\ w \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \\ E & F \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}
$$

where y is the measurement, z is the to-be-estimated variable, and w is the disturbance. Compute the minimally achievable H_2-norm of $w \to e$ and a corresponding observer for this system.

Exercise 6

Suppose the LMI region

$$
L := \left\{ z \in \mathbb{C} : \ Q + zS + \bar{z}S^\top \prec 0 \right\}
$$

is given. For given matrices A and B, consider the problem of finding a state-feedback gain F such that $A + BF$ has all its eigenvalues in L.

(a) Does the LMI-test for $\lambda(A + BF) \subset L$ result in an inequality that is convex/an LMI in both F and K?

(b) Can you linearize/convexify the synthesis problem by a suitable variable transformation?

(c) Consider a general LMI region $\mathscr{L}_P$ as defined in the lectures. How can we verify whether $\mathscr{L}_P$ defines a convex set in $\mathbb{C}$?

Exercise 7 (Multi-objective and mixed control synthesis)

This is a simulation exercise that involves the synthesis of an active controller for the suspension

system in Exercise 3 of Chapter 2. We consider the rear wheel of a vehicle as is depicted in Figure 4.3. Here m_1 represents tire, wheel and rear axle mass, m_2 denotes a fraction of the semitrailer
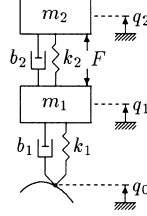

Figure 4.3: Active suspension system

mass. The deflection variables q_i are properly scaled so that $q_2 - q_1 = 0$ and $q_1 - q_0 = 0$ in steady state. The system is modeled by the state space equations

$$\begin{pmatrix} \dot{x} \\ z \end{pmatrix} = \begin{pmatrix} A & B_1 & B \\ C_1 & D_1 & E \end{pmatrix} \begin{pmatrix} x \\ q_0 \\ f \end{pmatrix}$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & -\frac{b_1+b_2}{m_1} & \frac{b_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & \frac{b_2}{m_2} & -\frac{b_2}{m_2} \end{pmatrix}; \quad B_1 = \begin{pmatrix} \frac{b_1}{m_1} \\ 0 \\ \frac{k_1}{m_1} - \frac{b_1}{m_1}\frac{b_1+b_2}{m_1} \\ \frac{b_1 b_2}{m_1 m_2} \end{pmatrix}; \quad B = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{m_1} \\ \frac{1}{m_2} \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & \frac{b_2}{m_2} & -\frac{b_2}{m_2} \\ -1 & 1 & 0 & 0 \end{pmatrix}; \quad D_1 = \begin{pmatrix} -1 \\ 0 \\ \frac{b_1 b_2}{m_1 m_2} \\ 0 \end{pmatrix}; \quad E = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{m_2} \\ 0 \end{pmatrix}$$

Here, $x = \mathrm{col}(q_1, q_2, \dot{q}_1 - b_1 q_0/m_1, \dot{q}_2)$ and $z = \mathrm{col}(q_1 - q_0, f, \ddot{q}_2, q_2 - q_1)$ define the state and the to-be-controlled output, respectively. The control input is the force f, the exogenous input is the road profile q_0.

Let the physical parameters be specified as in Table 2.1 in Chapter 2 and let $b_1 = 50$ [Ns/m]. The aim is to design an active suspension control system that generates the force f as a (causal) function of the measured variable $y = \mathrm{col}(\ddot{q}_2, q_2 - q_1)$. We wish to achieve low levels of acceleration throughout the vehicle ($\ddot{q}_2$), bounded suspension deflection ($q_2 - q_1$ and $q_1 - q_0$) and bounded dynamic tire force (f).

(a) Let the road profile be represented by $q_0 = W_{q_0} \tilde{q}_0$ where $\tilde{q}_0 \in \mathscr{L}_2$ is equalized in frequency and where

$$W_{q_0}(s) = \frac{0.01}{0.4s + 1}$$

reflects the quality of the road when the vehicle drives at constant speed. Define the to-be-controlled output $\tilde{z} = W_z z$ where W_z is a weighting matrix with transfer function

$$W_z(s) = \mathrm{diag}(200, 0.1, \frac{0.03s + 0.4}{0.000316s^2 + 0.0314s + 1}, 100).$$

Here, the dynamic weight on the chassis acceleration reflects the frequency band of the human sensitivity to vertical accelerations. Implement the generalized plant

$$P : \begin{pmatrix} \tilde{q}_0 \\ f \end{pmatrix} \mapsto \begin{pmatrix} \tilde{z} \\ y \end{pmatrix}$$

and synthesize a controller which minimizes the H_∞ norm of the closed-loop transfer function $\mathscr{T} : \tilde{q}_0 \mapsto \tilde{z}$.

(b) Construct the closed-loop system which maps q_0 to z (not $\tilde{q}_0$ to $\tilde{z}$!) and validate the controlled system by plotting the four frequency responses of the closed-loop system and the four responses to a road bump with amplitude 0.2 (meter). What are your conclusions about the behavior of this active suspension system?

(c) Partition the output z of the system into

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} ; \quad z_1 = \begin{pmatrix} q_1 - q_0 \\ f \end{pmatrix} ; \quad z_2 = \begin{pmatrix} \ddot{q}_2 \\ q_2 - q_1 \end{pmatrix}.$$

and let the weights on the signal components be as in the first part of this exercise. Let $\mathscr{T}_i$, $i = 1, 2$ be the transfer function mapping $\tilde{q}_0 \mapsto \tilde{z}_i$. We wish to obtain insight in the achievable trade-offs between upper bounds of $\|\mathscr{T}_1\|_\infty$ and $\|\mathscr{T}_2\|_2$. To do this,

   (i) Calculate the minimal achievable H_∞ norm of $\mathscr{T}_1$.
   (ii) Calculate the minimal achievable H_2 norm of $\mathscr{T}_2$.
   (iii) Calculate the minimal achievable H_2 norm of $\mathscr{T}_2$ subject to the bound $\|\mathscr{T}_1\|_\infty < \gamma_1$ where γ_1 takes some (≈ 5) values in the interval $[0.15, 0.30]$.

Make a plot of the Pareto optimal performances, i.e, plot the minimal achievable H_2 norm of $\mathscr{T}_2$ as function of γ_1.

Chapter 5

Robust stability and robust performance

First principle models of physical systems are often represented by state space equations in which components of the state variable represent well defined physical quantities. Variations, perturbations or uncertainties in specific physical parameters lead to uncertainty in the model. Often, this uncertainty is reflected by variations in well distinguished parameters or coefficients in the model, while the nature and/or range of the uncertain parameters may be known, or partially known. Since very small parameter variations may have a major impact on the dynamics of a system, it is of evident importance to analyse parametric uncertainties of dynamical systems. This will be the subject of this chapter. We reconsider the notions of nominal stability and nominal performance introduced in Chapter 3 in the light of different types of parametric uncertainty that effect the behavior of the system. We aim to derive robust stability tests and robust performance tests for systems with time-varying and rate-bounded parametric uncertainties.

5.1   Parametric uncertainties

Suppose that $\delta = (\delta_1, \ldots, \delta_p)$ is the vector which expresses the ensemble of all uncertain quantities in a given dynamical system. There are at least two distinct cases which are of independent interest:

(a) **time-invariant parametric uncertainties:** the vector δ is a fixed but unknown element of an uncertainty set $\boldsymbol{\delta} \subseteq \mathbb{R}^p$.

(b) **time-varying parametric uncertainties:** the vector δ is an unknown time varying function $\delta : \mathbb{R} \to \mathbb{R}^p$ whose values $\delta(t)$ belong to an uncertainty set $\boldsymbol{\delta} \subseteq \mathbb{R}^p$, and possibly satisfy additional constraints on rates of variation, continuity, spectral content, etc.

The first case typically occurs in models for which the physical parameters are fixed but only approximately known up to some level of accuracy. The second case typically captures models in which uncertain parameters, coefficients, or other physical quantities are time-dependent. One may object that in many practical situations both time-varying and time-invariant uncertainties occur so that the distinction between the two cases may seem somewhat artificial. This is true, but since time-invariant uncertainties can equivalently be viewed as time-varying uncertainties with a zero rate constraint, combined time-varying and time-invariant uncertainties are certainly not excluded.

A rather general class of *uncertain* continuous time, dynamical systems S is described by the state space equations

$$\dot{x} = f(x, w, \delta), \quad x(0) = x_0 \tag{5.1.1a}$$

$$z = g(x, w, \delta) \tag{5.1.1b}$$

where δ may or may not be time-varying, x, w and z are the state, input and output which take values in the state space X, the input space W and the output space Z, respectively. This constitutes a generalization of the model described in (2.2.1) defined in Chapter 2. If the uncertainties are fixed but unknown elements of an uncertainty set $\boldsymbol{\delta} \subseteq \mathbb{R}^p$ then one way to think of equations of this sort is to view S as a set of time-invariant systems, parametrized by $\delta \in \boldsymbol{\delta}$. However, if δ is time-dependent, then (5.1.1a) is to be interpreted as $\dot{x}(t) = f(x(t), w(t), \delta(t))$ and S is better viewed as a time-varying dynamical system. If the components of $\delta(t)$ coincide, for example, with state components then S may become a non-linear system, even when the mappings f and g are linear. If $\delta(t)$ is scalar valued and assumes values in a finite set $\boldsymbol{\delta} = \{1, \dots, K\}$ then S defines a hybrid system of K modes whose kth mode is defined by the dynamics

$$\dot{x} = f_k(x, w) := f(x, w, k), \quad x(0) = x_0$$
$$z = g_k(x, w) := g(x, w, k)$$

and where the time-varying behavior of $\delta(t)$ defines the switching events between the various modes. In any case, the system S defined by (5.1.1) is of considerable theoretical and practical interest as it covers quite some relevant classes of dynamical systems.

5.1.1 Affine parameter dependent systems

If f and g in (5.1.1) are linear in x and w then the uncertain system S assumes a representation

$$\begin{cases} \dot{x} = A(\delta)x + B(\delta)w, & x(0) = x_0 \\ z = C(\delta)x + D(\delta)w \end{cases} \tag{5.1.2}$$

in which δ may or may not be time-varying.

Of particular interest will be those systems (5.1.2) in which the matrices *affinely* depend on δ. This means that

$$S(\delta) := \begin{pmatrix} A(\delta) & B(\delta) \\ C(\delta) & D(\delta) \end{pmatrix} = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} + \delta_1 \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} + \dots + \delta_p \begin{pmatrix} A_p & B_p \\ C_p & D_p \end{pmatrix}$$

or, written in more compact form, $S(\delta) = S_0 + \delta_1 S_1 + \ldots + \delta_p S_p$ where

$$S_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}, \quad i = 1, \ldots, p$$

are the coefficient matrices associated with (5.1.2). Models of this type are referred to as *affine parameter dependent models*.

5.1.2 Polytopic parameter dependent systems

As an alternative representation of uncertainty, consider the system (5.1.2) and suppose that any matrix

$$S(\delta) = \begin{pmatrix} A(\delta) & B(\delta) \\ C(\delta) & D(\delta) \end{pmatrix} \quad \text{with } \delta \in \boldsymbol{\delta}$$

can be written as the *convex combination* of N system matrices

$$S^j = \begin{pmatrix} A^j & B^j \\ C^j & D^j \end{pmatrix}, \quad j = 1, \ldots, N.$$

(where j is a superscript, not a power). This means that for any $\delta \in \boldsymbol{\delta}$ there exist $\alpha_j \geq 0$ with $\sum_{j=1}^N \alpha_j = 1$ such that

$$S(\delta) = \sum_{j=1}^N \alpha_j S^j.$$

Stated otherwise, $\{S(\delta) \mid \delta \in \boldsymbol{\delta}\} = \text{conv}(S^1, \ldots, S^N)$. Models of this type are referred to as *polytopic linear differential inclusions*.

It is easily seen that a polytopic linear differential inclusion is a special cases of an affine parameter dependent system if the uncertainty set $\boldsymbol{\delta}$ is polytopic and equal to the convex hull $\boldsymbol{\delta} = \text{conv}(\boldsymbol{\delta}_g)$ where $\boldsymbol{\delta}_g = \{\delta^1, \ldots, \delta^N\}$ is a finite set. Indeed, in that case we have

$$\{S(\delta) \mid \delta \in \boldsymbol{\delta}\} = \{S(\delta) \mid \delta \in \text{conv}(\boldsymbol{\delta}_g)\} = \text{conv}\{S^1, \ldots, S^N\}$$

where $S^j = S(\delta^j)$ define the generator matrices of the polytopic model.

5.2 Robust stability

5.2.1 Time-invariant parametric uncertainty

An important issue in the design of control systems involves the question as to what extend the stability and performance of the controlled system is robust against perturbations and uncertainties

in the parameters of the system. In this section we consider the uncertain time-invariant system defined by

$$\dot{x} = A(\delta)x \tag{5.2.1}$$

where the state matrix $A(\cdot)$ is a continuous function of a real valued time-invariant parameter vector $\delta = \mathrm{col}(\delta_1, \ldots, \delta_p)$ which we assume to be contained in an *uncertainty set* $\boldsymbol{\delta} \subseteq \mathbb{R}^p$. Let $X = \mathbb{R}^n$ be the state space of this system. We will analyze the *robust stability* of the equilibrium point $x^* = 0$ of this system. Precisely, we address the question when the equilibrium point $x^* = 0$ of (5.2.1) is asymptotically stable in the sense of Definition 3.1 for all $\delta \in \boldsymbol{\delta}$.

Example 5.1 As an example, let

$$A(\delta) = \begin{pmatrix} -1 & 2\delta_1 & 2 \\ \delta_2 & -2 & 1 \\ 3 & -1 & \frac{\delta_3 - 10}{\delta_1 + 1} \end{pmatrix}$$

where $\delta_1 \in [-0.5, 1]$, $\delta_2 \in [-2, 1]$ and $\delta_3 \in [-0.5, 2]$. Then the uncertainty set $\boldsymbol{\delta}$ is polytopic and defined by

$$\boldsymbol{\delta} = \{\mathrm{col}(\delta_1, \delta_2, \delta_3) \mid \delta_1 \in [-0.5, 1], \ \delta_2 \in [-2, 1], \ \delta_3 \in [-0.5, 2]\}.$$

Moreover, $\boldsymbol{\delta} = \mathrm{conv}(\boldsymbol{\delta}_{\mathrm{g}})$ with

$$\boldsymbol{\delta}_{\mathrm{g}} = \{\mathrm{col}(\delta_1, \delta_2, \delta_3) \mid \delta_1 \in \{-0.5, 1\}, \ \delta_2 \in \{-2, 1\}, \ \delta_3 \in \{-0.5, 2\}\}$$

the set of vertices (or generators) of $\boldsymbol{\delta}$.

For time-invariant parametric uncertainties, the system $\dot{x} = A(\delta)x$ is asymptotically stable if and only if $A(\delta)$ is Hurwitz for all $\delta \in \boldsymbol{\delta}$. That is, if and only if the eigenvalues of $A(\delta)$ lie in the open left-half complex plane for all admissible perturbations $\delta \in \boldsymbol{\delta}$. Hence, using Proposition 3.6, the verification of robust stability amounts to checking whether

$$\rho(A(\delta)) := \max \mathrm{Re}(\lambda(A(\delta))) < 0 \quad \text{for all } \delta \in \boldsymbol{\delta}.$$

There are at least three reasons why this is difficult to verify. Firstly, if $\boldsymbol{\delta}$ is a continuum in $\mathbb{R}^p$, this means verifying an inequality at an infinite number of points. Secondly, if $\boldsymbol{\delta}$ is a polytope, it will generally *not suffice* to check the above inequality on the vertices of the uncertainty set only. Thirdly, since $\rho(A(\delta))$ is, in general, not a convex or concave function of δ it will be numerically difficult to find the global maximum over $\boldsymbol{\delta}$.

Quadratic stability with time-invariant uncertainties

We will apply Theorem 3.4 to infer the asymptotic stability of the equilibrium point $x^* = 0$ of (5.2.1).

Definition 5.2 (Quadratic stability) The system (5.2.1) is said to be *quadratically stable for perturbations* $\delta \in \boldsymbol{\delta}$ if there exists a matrix $K = K^\top$ such that

$$K \succ 0 \quad \text{and} \quad A(\delta)^\top K + KA(\delta) \prec 0 \quad \text{for all } \delta \in \boldsymbol{\delta}. \tag{5.2.2}$$

The importance of this definition becomes apparent when considering quadratic Lyapunov functions $V(x) = x^\top K x$. Indeed, if K satisfies (5.2.2) then there exists an $\varepsilon > 0$ such that

$$A(\delta)^\top K + K A(\delta) + \varepsilon K \preccurlyeq 0 \quad \text{for all } \delta \in \boldsymbol{\delta}.$$

The time-derivative $\dot{\overline{V}}$ of the composite function $\overline{V}(t) := V(x(t)) = x(t)^\top K x(t)$ along solutions of (5.2.1), defined in (3.1.5), then satisfies

$$\dot{\overline{V}}(t) + \varepsilon\overline{V}(t) = \partial_x V(x(t)) A(\delta) x(t) + \varepsilon V(x(t))$$
$$= x^\top(t) \left[A(\delta)^\top K + K A(\delta) + \varepsilon K \right] x(t) \leq 0$$

for all $t \geq 0$ and all $\delta \in \boldsymbol{\delta}$. Integrating this expression over an interval $[t_0, t_1]$ shows that $\overline{V}$ has exponential decay according to $\overline{V}(t_1) \leq \overline{V}(t_0) e^{-\varepsilon(t_1 - t_0)}$ for all $t_1 \geq t_0$ and all $\delta \in \boldsymbol{\delta}$. Now use that

$$\lambda_{\min}(K) \|x\|^2 \leq x^\top K x \leq \lambda_{\max}(K) \|x\|^2$$

to see that

$$\|x(t_1)\|^2 \leq \frac{1}{\lambda_{\min}(K)} V(x(t_1)) \leq \frac{1}{\lambda_{\min}(K)} V(x(t_0)) e^{-\varepsilon(t_1 - t_0)} \leq \frac{\lambda_{\max}(K)}{\lambda_{\min}(K)} \|x(t_0)\|^2 e^{-\varepsilon(t_1 - t_0)}.$$

That is,

$$\|x(t)\| \leq \|x(t_0)\| \sqrt{\frac{\lambda_{\max}(K)}{\lambda_{\min}(K)}} e^{-\frac{\varepsilon}{2}(t - t_0)} \quad \text{for all } \delta \in \boldsymbol{\delta}, \ t \geq 0.$$

Conclude that the origin of a system that is quadratically stable is globally exponentially stable (and hence globally asymptotically stable) for all perturbations $\delta \in \boldsymbol{\delta}$. In particular, the exponential decay rate, $\varepsilon/2$ does not depend on δ.

It is truly worthwhile to understand (better even to appreciate) the arguments in this reasoning as they are at the basis of more general results to come.

Verifying quadratic stability with time-invariant uncertainties

By (5.2.2), the verification of quadratic stability of a system places an infinite number of constraints on the symmetric matrix K if $\boldsymbol{\delta}$ is a continuum of uncertain parameters. It is the purpose of this section to make additional assumptions on the way the uncertainty enters the system, so as to convert (5.2.2) into a numerically tractable condition.

Suppose that $A(\delta)$ is an *affine* function of the parameter vector δ, i.e.,

$$A(\delta) = A_0 + \delta_1 A_1 + \cdots + \delta_p A_p \tag{5.2.3}$$

for some real matrices $A_0, \ldots, A_p$, all of dimension $n \times n$. In addition, let us suppose that the uncertainty set $\boldsymbol{\delta}$ is convex and coincides with the convex hull of a set $\boldsymbol{\delta}_{\mathrm{g}} \subset \mathbb{R}^p$. With this structure on A and $\boldsymbol{\delta}$ we have the following result.

Proposition 5.3 *If $A(\cdot)$ is an affine function and $\boldsymbol{\delta} = \mathrm{conv}(\boldsymbol{\delta}_g)$ with $\boldsymbol{\delta}_g \subset \mathbb{R}^p$, then the system (5.2.1) is quadratically stable if and only if there exists $K = K^\top$ such that*

$$K \succ 0 \quad and \quad A(\delta)^\top K + KA(\delta) \prec 0 \quad for\ all\ \delta \in \boldsymbol{\delta}_g.$$

Proof. The proof of this result is an application of Proposition 1.14 in Chapter 1. Indeed, consider the mapping $f : \boldsymbol{\delta} \to \mathbb{S}$ defined by

$$f(\delta) := A(\delta)^\top K + KA(\delta).$$

Then the domain $\boldsymbol{\delta} = \mathrm{conv}(\boldsymbol{\delta}_g)$ of f is convex. Since $A(\cdot)$ is affine, it follows that f is a convex function. By Proposition 1.14 (Chapter 1) it follows that $f(\delta) \prec 0$ for all $\delta \in \boldsymbol{\delta}$ if and only if $f(\delta) \prec 0$ for all $\delta \in \boldsymbol{\delta}_g$. Hence, $A(\delta)^\top K + KA(\delta) \prec 0$ for all $\delta \in \boldsymbol{\delta}$ if and only if the same inequality holds for all $\delta \in \boldsymbol{\delta}_g$. This yields the result. ∎

Obviously, the importance of this result lies in the fact that quadratic stability can be concluded from a *finite test* of matrix inequalities whenever $\boldsymbol{\delta}_g$ consists of a finite number of elements. That is, when the uncertainty set is the convex hull of a finite number of points in $\mathbb{R}^p$. In that case, the condition stated in Proposition 5.3 is a feasibility test of a (finite) system of LMI's.

Example 5.4 Continuing Example 5.1, the matrix $A(\delta)$ is not affine in δ, but by setting $\delta_4 = \frac{\delta_3 - 10}{\delta_1 + 1} + 12$ we obtain that

$$A(\delta) = \begin{pmatrix} -1 & 2\delta_1 & 2 \\ \delta_2 & -2 & 1 \\ 3 & -1 & \delta_4 - 12 \end{pmatrix}, \quad \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_4 \end{pmatrix} \in \boldsymbol{\delta} = [-0.5, 1] \times [-2, 1] \times [-9, 8].$$

covers the set of matrices in Example 5.1. Since $\boldsymbol{\delta} = \mathrm{conv}(\boldsymbol{\delta}_g)$ with $\boldsymbol{\delta}_g = \{\delta^1, \ldots, \delta^N\}$ consisting of the $N = 2^3 = 8$ vertices of the uncertainty set, the verification of the quadratic stability of (5.2.1) is a feasibility test of the 9 LMI's

$$K \succ 0, \qquad A(\delta^j)^\top K + KA(\delta^j) \prec 0, \quad j = 1, \ldots, 8.$$

The test will not pass. By Proposition 5.3, the system is not quadratically stable for the given uncertainty set.

Example 5.5 Consider the uncertain control system $\dot{x} = A(\delta)x + B(\delta)u$ where we wish to construct a feedback law $u = Fx$ such that the controlled system $\dot{x} = (A(\delta) + B(\delta)F)x$ is quadratically stable for all δ in some polytopic uncertainty set $\boldsymbol{\delta} = \mathrm{conv}(\boldsymbol{\delta}_g)$ with $\boldsymbol{\delta}_g$ the set of vertices. By Proposition 5.3, this is equivalent to finding F and $K = K^\top$ such that

$$K \succ 0 \quad and \quad (A(\delta) + B(\delta)F)^\top K + K(A(\delta) + B(\delta)F) \prec 0, \quad for\ all \quad \delta \in \boldsymbol{\delta}_g$$

This is not a system of LMI's. However, with $X = K^{-1}$ and $L = FK^{-1}$ and assuming that $A(\cdot)$ and $B(\cdot)$ are affine, we can transform the latter to an LMI feasibility test to find $X = X^\top$ and L such that

$$X \succ 0 \quad and \quad A(\delta)X + XA(\delta) + B(\delta)L + (B(\delta)L)^\top \prec 0, \quad for\ all \quad \delta \in \boldsymbol{\delta}_g.$$

Whenever the test passes, the quadratically stabilizing feedback law is given by $F = LX^{-1}$.

Parameter-dependent Lyapunov functions with time-invariant uncertainties

The main disadvantage in searching for one quadratic Lyapunov function for a class of uncertain models is the conservatism of the test. Indeed, Example 5.1 shows that (5.2.1) may not be quadratically stable, but no conclusions can been drawn from this observation concerning the stability of the uncertain system. To reduce conservatism of the quadratic stability test we will consider quadratic Lyapunov functions for the system (5.2.1) which are *parameter dependent*, i.e., Lyapunov functions $V : X \times \boldsymbol{\delta} \to \mathbb{R}$ of the form

$$V(x, \delta) := x^\top K(\delta) x$$

where $K(\delta)$ is a matrix valued function that is allowed to depend on the uncertain parameter δ. A sufficient condition for robust asymptotic stability can be stated as follows.

Proposition 5.6 *Let the uncertainty set $\boldsymbol{\delta}$ be compact and suppose that $K(\delta)$ is continuously differentiable on $\boldsymbol{\delta}$ and satisfies*

$$K(\delta) \succ 0 \qquad \qquad \text{for all } \delta \in \boldsymbol{\delta} \tag{5.2.4a}$$

$$A(\delta)^\top K(\delta) + K(\delta) A(\delta) \prec 0 \quad \text{for all } \delta \in \boldsymbol{\delta}. \tag{5.2.4b}$$

Then the system (5.2.1) *is globally asymptotically stable for all $\delta \in \boldsymbol{\delta}$.*

Proof.   Let $K(\delta)$ satisfy (5.2.4) and consider $V(x, \delta) = x^\top K(\delta) x$ as candidate Lyapunov function. There exists $\varepsilon > 0$ such that $K(\delta)$ satisfies $A(\delta)^\top K(\delta) + K(\delta) A(\delta) + \varepsilon K(\delta) \preccurlyeq 0$. Take the time derivative of the composite function $\overline{V}(t) := V(x(t), \delta)$ along solutions of (5.2.1) to infer that $\dot{\overline{V}}(t) + \varepsilon \overline{V}(t) \leq 0$ for all $t \in \mathbb{R}$ and all $\delta \in \boldsymbol{\delta}$. This means that for all $\delta \in \boldsymbol{\delta}$, $\overline{V}(\cdot)$ is exponentially decaying along solutions of (5.2.1) according to $\overline{V}(t) \leq \overline{V}(0) e^{-\varepsilon t}$. Define $a := \inf_{\delta \in \boldsymbol{\delta}} \lambda_{\min}(K(\delta))$ and $b := \sup_{\delta \in \boldsymbol{\delta}} \lambda_{\max}(K(\delta))$. If $\boldsymbol{\delta}$ is compact, the positive definiteness of K implies that both a and b are positive and we have that $a \|x\|^2 \leq V(x, \delta) \leq b \|x\|^2$ for all $\delta \in \boldsymbol{\delta}$ and all $x \in X$. Together with the exponential decay of $\overline{V}$ this yields that $\|x(t)\|^2 \leq \frac{b}{a} \|x(0)\|^2 e^{-\varepsilon t}$ for all $\delta \in \boldsymbol{\delta}$ which proves the exponential and asymptotic stability of (5.2.1). $\blacksquare$

The search for matrix valued functions that satisfy the conditions (5.2.4) is much more involved and virtually intractable from a computational point of view. There are many ways to turn Proposition 5.6 into a numerically efficient scheme that yield parameter varying Lyapunov functions. Here is one: consider Lyapunov functions that are *affine* in the parameter δ, i.e.,

$$K(\delta) = K_0 + \delta_1 K_1 + \ldots + \delta_p K_p$$

where $K_0, \ldots, K_p$ are real symmetric matrices of dimension $n \times n$ and $\delta = \text{col}(\delta_1, \ldots, \delta_p)$ is the time-invariant uncertainty vector. Clearly, with $K_1 = \ldots = K_p = 0$ we are back to the case of parameter *independent* quadratic Lyapunov functions as discussed in the previous subsection. The system (5.2.1) is called *affine quadratically stable* if there exist matrices $K_0, \ldots, K_p$ such that $K(\delta)$ satisfies the conditions (5.2.4) of Proposition 5.6.

Let $A(\cdot)$ be affine and represented by (5.2.3). Suppose that $\boldsymbol{\delta}$ is convex with $\boldsymbol{\delta} = \mathrm{conv}(\boldsymbol{\delta}_\mathrm{g})$ where $\boldsymbol{\delta}_\mathrm{g}$ is a finite set of vertices of $\boldsymbol{\delta}$. Then the expression

$$L(\delta) := A(\delta)^\top K(\delta) + K(\delta)A(\delta)$$

in (5.2.4b) is, in general, not affine in δ. As a consequence, the function $f : \boldsymbol{\delta} \to \mathbb{S}$ defined by

$$f(\delta) := L(\delta) \tag{5.2.5}$$

will not be convex so that the implication

$$\{f(\delta) \prec 0 \text{ for all } \delta \in \boldsymbol{\delta}_\mathrm{g}\} \quad \Longrightarrow \quad \{f(\delta) \prec 0 \text{ for all } \delta \in \boldsymbol{\delta}\} \tag{5.2.6}$$

used in the previous section (proof of Proposition 5.3), will not hold. Expanding $L(\delta)$ yields

$$L(\delta) = [A_0 + \sum_{j=1}^p \delta_j A_j]^\top [K_0 + \sum_{j=1}^p \delta_j K_j] + [K_0 + \sum_{j=1}^p \delta_j K_j][A_0 + \sum_{j=1}^p \delta_j A_j]$$

$$= \sum_{i=0}^p \sum_{j=0}^p \delta_i \delta_j [A_i^\top K_j + K_j A_i]$$

where, to allow compact notation, we set $\delta_0 = 1$. Consequently, (5.2.5) takes the form

$$f(\delta) = C_0 + \sum_{j=1}^p \delta_j C_j + \sum_{j=1}^p \sum_{i=1}^{j-1} \delta_i \delta_j C_{ij} + \sum_{j=1}^p \delta_j^2 D_j$$

where C_0, C_j, C_{ij} and D_j are symmetric matrices. Now consider the following "box-shaped" uncertainty sets

$$\boldsymbol{\delta} = \{\delta \in \mathbb{R}^p \mid \delta_k \in [\underline{\delta}_k, \bar{\delta}_k] \}, \qquad \boldsymbol{\delta}_\mathrm{g} = \{\delta \in \mathbb{R}^p \mid \delta_k \in \{\underline{\delta}_k, \bar{\delta}_k\} \} \tag{5.2.7}$$

Here, $\underline{\delta}_k \leq \bar{\delta}_k$ and it is immediate that $\boldsymbol{\delta} = \mathrm{conv}(\boldsymbol{\delta}_\mathrm{g})$. It is easily seen that a sufficient condition for the implication (5.2.6) to hold for the uncertainty sets (5.2.7) is that $f(\delta_1, \ldots, \delta_j, \ldots, \delta_p)$ is *partially convex*, that is f is convex in each of its arguments δ_j, $j = 1, \ldots, p$ separately. Since f is a twice differentiable function, f is partially convex if

$$D_j = \frac{1}{2} \frac{\partial^2 f}{\partial \delta_j^2} = A_j^\top K_j + K_j A_j \succcurlyeq 0$$

for $j = 1, \ldots, p$. We therefore obtain that

$$A_j^\top K_j + K_j A_j \succcurlyeq 0, \qquad j = 1, \ldots, p$$

is a sufficient condition for (5.2.6) to hold on the uncertainty sets (5.2.7). This leads to the following main result.

Theorem 5.7 *If $A(\cdot)$ is an affine function described by (5.2.3) and $\boldsymbol{\delta} = \mathrm{conv}(\boldsymbol{\delta}_g)$ assumes the form (5.2.7), then the system (5.2.1) is affine quadratically stable if there exist real matrices $K_0, \ldots, K_p$*

such that $K(\delta) = K_0 + \sum_{j=1}^{p} \delta_j K_j$ *satisfies*

$$A(\delta)^\top K(\delta) + K(\delta)A(\delta) \prec 0 \quad \textit{for all } \delta \in \boldsymbol{\delta}_g \tag{5.2.8a}$$

$$K(\delta) \succ 0 \quad \textit{for all } \delta \in \boldsymbol{\delta}_g \tag{5.2.8b}$$

$$A_j^\top K_j + K_j A_j \succcurlyeq 0 \quad \textit{for } j = 1, \ldots, p. \tag{5.2.8c}$$

In that case, the parameter varying function satisfies the conditions (5.2.4) *and* $V(x, \delta) := x^\top K(\delta)x$ *is a quadratic parameter-dependent Lyapunov function of the system.*

Proof. It suffices to prove that (5.2.8) implies (5.2.4). Since $K(\delta)$ is affine in δ, the mapping

$$\delta \mapsto -K(\delta)$$

with $\delta \in \boldsymbol{\delta}$ is convex. Consequently, $-K(\delta) \preccurlyeq 0$ for all $\delta \in \boldsymbol{\delta}$ if and only if $-K(\delta) \preccurlyeq 0$ for all $\delta \in \boldsymbol{\delta}_g$. This yields that (5.2.4a) is implied by (5.2.8b). The proof that (5.2.8a) and (5.2.8c) imply (5.2.4b) was given in the arguments preceding this theorem. ∎

Theorem 5.7 reduces the problem to verify affine quadratic stability of the system (5.2.1) with box-type uncertainties to a feasibility problem of a (finite) set of linear matrix inequalities.

5.2.2 Time-varying parametric uncertainty

Robust stability against time-varying perturbations is generally a more demanding requirement than robust stability against time-invariant parameter uncertainties. In this section we consider the question of robust stability for the system

$$\dot{x}(t) = A(\delta(t))x(t) \tag{5.2.9}$$

where the values of the time-varying parameter vector $\delta(t)$ belong to the uncertainty set $\boldsymbol{\delta} \subset \mathbb{R}^p$ for all time $t \in \mathbb{R}$. In this section we assess the *robust stability* of the fixed point $x^* = 0$. It is important to remark that, unlike the case with time-invariant uncertainties, robust stability of the origin of the time-varying system (5.2.9) is *not equivalent* to the condition that the (time-varying) eigenvalues $\lambda(A(\delta(t)))$ belong to the stability region $\mathbb{C}^-$ for all admissible perturbations $\delta(t) \in \boldsymbol{\delta}$.

Proposition 5.8 *The uncertain system* (5.2.9) *with time-varying uncertainties* $\delta(\cdot) \in \boldsymbol{\delta}$ *is asymptotically stable if there exists a matrix* $K = K^\top$ *such that* (5.2.2) *holds.*

The inequalities (5.2.2) are therefore sufficient conditions to conclude asymptotic stability of the *time-varying* uncertainties that occur in (5.2.9). Since Proposition 5.8 is obtained as a special case of Theorem 5.10 below, we defer its proof. An interesting observation related to Proposition 5.8 is that the existence of a real symmetric matrix K satisfying (5.2.2) not only yields quadratic stability of the system (5.2.1) with $\delta \in \boldsymbol{\delta}$ but also the asymptotic stability of (5.2.9) with $\delta(t) \in \boldsymbol{\delta}$. Hence, the

existence of such K implies that *arbitrary fast variations* in the time-varying parameter vector $\delta(\cdot)$ may occur so as to guarantee asymptotic stability of (5.2.9). If additional *a priori* information on the time-varying parameters is known, the result of Proposition 5.8 may become too conservative and we therefore may like to resort to different techniques to incorporate information about the parameter trajectories $\delta(\cdot)$.

A rather natural way to incorporate additional a priori information on time-varying uncertainties is to assume that trajectories $\delta(\cdot)$ are both value and rate constrained.

Assumption 5.9 The trajectories $\delta(\cdot)$ are continuously differentiable and satisfy

$$\delta(t) \in \boldsymbol{\delta}, \quad \dot{\delta}(t) \in \boldsymbol{\lambda} \qquad \text{for all time } t \in \mathbb{R}.$$

where $\boldsymbol{\delta}$ and $\boldsymbol{\lambda}$ are compact subsets of $\mathbb{R}^p$.

We will therefore assume that not only the *values* but also the *rates* of the parameter trajectories are constrained.

Quadratic stability with time-varying uncertainties

A central result for achieving robust stability of the system (5.2.9) against all uncertainties that satisfy Assumption 5.9 is given in the following theorem.

Theorem 5.10 *Suppose that the function $K : \boldsymbol{\delta} \to \mathbb{S}^n$ is continuously differentiable on a compact set $\boldsymbol{\delta}$ and satisfies*

$$K(\delta) \succ 0 \quad \text{for all } \delta \in \boldsymbol{\delta} \tag{5.2.10a}$$

$$\partial_\delta K(\delta)\lambda + A(\delta)^\top K(\delta) + K(\delta)A(\delta) \prec 0 \quad \text{for all } \delta \in \boldsymbol{\delta} \text{ and } \lambda \in \boldsymbol{\lambda} \tag{5.2.10b}$$

Then the origin of the system (5.2.9) *is exponentially stable against all time-varying uncertainties $\delta : \mathbb{R} \to \mathbb{R}^p$ that satisfy Assumption 5.9. Moreover, in that case $V(x, \delta) := x^\top K(\delta)x$ is a quadratic parameter depending Lyapunov function for the system* (5.2.9).

Note that this result involves taking partial derivatives of the matrix functions $K(\cdot)$. Specifically, in (5.2.10b), we have used the compact notation

$$\partial_\delta K(\delta)\lambda = \frac{\partial K}{\partial \delta_1}(\delta)\lambda_1 + \ldots + \frac{\partial K}{\partial \delta_p}(\delta)\lambda_p = \sum_{k=1}^{p} \frac{\partial K}{\partial \delta_k}(\delta)\lambda_k$$

for the partial derivatives of matrix functions. To further simplify notation, let $\partial_k := \frac{\partial}{\partial \delta_k}$ denote the operator that takes partial derivatives with respect to the kth entry in δ. (You need good eyes to distinguish the symbols).

Proof. The proof follows very much the same lines as the proof of Proposition 5.6, but now includes time-dependence of the parameter functions. Suppose that $K(\delta)$ satisfies the hypothesis. Consider $V(x, \delta) = x^\top K(\delta)x$ as candidate Lyapunov function. Let $a := \inf_{\delta \in \boldsymbol{\delta}} \lambda_{\min}K(\delta)$ and $b := \sup_{\delta \in \boldsymbol{\delta}} \lambda_{\max}K(\delta)$. If $\boldsymbol{\delta}$ is compact, the positive definiteness of $K(\delta)$ for all $\delta \in \boldsymbol{\delta}$ implies that both a and b are positive. In addition, we can find $\varepsilon > 0$ such that $K(\delta)$ satisfies

$$aI \preccurlyeq K(\delta) \preccurlyeq bI, \quad \partial_\delta K(\delta)\lambda + A(\delta)^\top K(\delta) + K(\delta)A(\delta) + \varepsilon K(\delta) \preccurlyeq 0$$

for all $\delta \in \boldsymbol{\delta}$ and $\lambda \in \boldsymbol{\lambda}$. Take the time derivative of the composite function $\overline{V}(t) := V(x(t), \delta(t))$ along solutions of (5.2.9) to infer that

$$\dot{\overline{V}}(t) + \varepsilon \overline{V}(t) = x(t)^\top A(\delta(t))^\top K(\delta(t))x(t) + x(t)^\top K(\delta(t))A(\delta(t))x(t) +$$
$$+ \varepsilon x(t)^\top K(\delta(t))x(t) + x(t)^\top \left\{ \sum_{k=1}^{p} \partial_k K(\delta(t))\dot{\delta}_k(t) \right\}x(t) \leq 0$$

for all $t \in \mathbb{R}$, all $\delta(t) \in \boldsymbol{\delta}$ and all $\dot{\delta}(t) \in \boldsymbol{\lambda}$. This means that for this class of uncertainties $\overline{V}$ is exponentially decaying along solutions of (5.2.1) according to $\overline{V}(t) \leq \overline{V}(0)e^{-\varepsilon t}$. Moreover, since $a\|x\|^2 \leq V(x, \delta) \leq b\|x\|^2$ for all $\delta \in \boldsymbol{\delta}$ and all $x \in X$ we infer that $\|x(t)\|^2 \leq \frac{b}{a}\|x(0)\|^2 e^{-\varepsilon t}$ for all $t \geq 0$ and all uncertainties $\delta(t)$ satisfying Assumption 5.9. Hence, (5.2.9) is exponentially stable against uncertainties that satisfy Assumption 5.9. $\blacksquare$

Theorem 5.10 involves a search for matrix functions satisfying the inequalities (5.2.10) to guarantee robust asymptotic stability. Note that the result is a sufficient algebraic test only that provides a quadratic parameter dependent Lyapunov function, when the test passes. The result is not easy to apply or verify by a computer program as it involves a partial differential equation on $K(\delta)$ and, in general, an infinite number of conditions on the inequalities (5.2.10). We will therefore focus on a number of special cases that convert Theorem 5.10 in a feasible numerical test.

For this, first consider the case where the parameters are time-invariant. This is equivalent to saying that $\boldsymbol{\lambda} = \{0\}$. The conditions (5.2.10) then coincide with (5.2.4) and we therefore obtain Proposition 5.6 as a special case. In particular, the sufficient condition (5.2.10) for robust stability in Theorem 5.10 is also necessary in this case.

If we assume arbitrary fast time-variations in $\delta(t)$ then we consider rate constraints of the form $\boldsymbol{\lambda} = [-r, r]^p$ with $r \to \infty$. For (5.2.10b) to hold for any λ with $|\lambda_k| > r$ and $r \to \infty$ it is immediate that $\partial_\delta K(\delta)$ needs to vanish for all $\delta \in \boldsymbol{\delta}$. Consequently, in this case K can not depend on δ and Theorem 5.10 reduces to Proposition 5.8. In particular, this argument proves Proposition 5.8 as a special case.

Verifying quadratic stability with time-varying uncertainties

In this section we will assume that $A(\cdot)$ in (5.2.9) is an affine function of $\delta(t)$. The uncertainty sets $\boldsymbol{\delta}$ and $\boldsymbol{\lambda}$ are assumed to be convex sets defined by the 'boxes'

$$\boldsymbol{\delta} = \{\delta \in \mathbb{R}^p \mid \delta_k \in [\underline{\delta}_k, \bar{\delta}_k]\}, \qquad \boldsymbol{\lambda} = \{\lambda \in \mathbb{R}^p \mid \lambda_k \in [\underline{\lambda}_k, \bar{\lambda}_k]\} \tag{5.2.11}$$

Stated otherwise, the uncertainty regions are the convex hulls of the sets

$$\boldsymbol{\delta}_{\mathrm{g}} = \{\delta \in \mathbb{R}^p \mid \delta_k \in \{\underline{\delta}_k, \bar{\delta}_k\}\}, \qquad \boldsymbol{\lambda}_{\mathrm{g}} = \{\lambda \in \mathbb{R}^p \mid \lambda_k \in \{\underline{\lambda}_k, \bar{\lambda}_k\}\}.$$

In addition, the search of a parameter dependent $K(\delta)$ will be restricted to the class of *affine functions* $K(\delta)$ represented by

$$K(\delta) = K_0 + \delta_1 K_1 + \cdots + \delta_p K_p$$

where $K_j \in \mathbb{S}^n$, $0 = 1, \ldots, p$ is symmetric. For this class of parameter dependent functions we have that $\partial_k K(\delta) = K_k$ so that (5.2.10b) reads

$$\sum_{k=1}^{p} K_k \lambda_k + \sum_{v=0}^{p} \sum_{\mu=0}^{p} \delta_v \delta_\mu (A_v^\top K_\mu + K_\mu A_v) \prec 0 \quad \text{for all } \delta_v, \delta_\mu \in \boldsymbol{\delta} \text{ and } \lambda_k \in \boldsymbol{\lambda}.$$

Here, we set $\delta_0 = 1$ to simplify notation. Now note that the latter expression is affine in $K_0, \ldots, K_p$, affine in $\lambda_1, \ldots, \lambda_p$ and quadratic in $\delta_1, \ldots, \delta_p$ due to the mixture of constant, linear and quadratic terms. Similar to (5.2.5), we introduce the function $f : \boldsymbol{\delta} \times \boldsymbol{\lambda} \to \mathbb{R}$ defined by

$$f(\delta, \lambda) := \sum_{k=1}^{p} K_k \lambda_k + \sum_{v=0}^{p} \sum_{\mu=0}^{p} \delta_v \delta_\mu (A_v^\top K_\mu + K_\mu A_v).$$

A sufficient condition for the implication

$$\{f(\delta, \lambda) \prec 0 \text{ for all } (\delta, \lambda) \in \boldsymbol{\delta}_{\mathrm{g}} \times \boldsymbol{\lambda}_{\mathrm{g}}\} \quad \Longrightarrow \quad \{f(\delta, \lambda) \prec 0 \text{ for all } (\delta, \lambda) \in \boldsymbol{\delta} \times \boldsymbol{\lambda}\}$$

is that f is *partially convex* in each of its arguments δ_j, $j = 1, \ldots, p$. As in subsection 5.2.1, this condition translates to the requirement that

$$A_j^\top K_j + K_j A_j \succcurlyeq 0, \qquad j = 1, \ldots, p,$$

which brings us to the following main result.

Theorem 5.11 *Suppose that $A(\cdot)$ is affine as described by (5.2.3), and assume that $\delta(t)$ satisfies Assumption 5.9 with $\boldsymbol{\delta}$ and $\boldsymbol{\lambda}$ compact box-shaped sets specified in (5.2.11). Then the origin of the system (5.2.9) is robustly asymptotically stable against all time-varying uncertainties that satisfy Assumption 5.9 if there exist real matrices $K_0, \ldots, K_p$ such that $K(\delta) = K_0 + \sum_{j=1}^{p} \delta_j K_j$ satisfies*

$$\sum_{k=1}^{p} K_k \lambda_k + \sum_{v=0}^{p} \sum_{\mu=0}^{p} \delta_v \delta_\mu \left(A_v^\top K_\mu + K_\mu A_v\right) \prec 0 \quad \textit{for all } \delta \in \boldsymbol{\delta}_g \textit{ and } \lambda \in \boldsymbol{\lambda}_g \qquad (5.2.12\mathrm{a})$$

$$K(\delta) \succ 0 \quad \textit{for all } \delta \in \boldsymbol{\delta}_g \qquad (5.2.12\mathrm{b})$$

$$A_j^\top K_j + K_j A_j \succcurlyeq 0 \quad \textit{for } j = 1, \ldots, p. \qquad (5.2.12\mathrm{c})$$

Moreover, in that case, $V(x, \delta) := x^\top K(\delta) x$ defines a quadratic parameter-dependent Lyapunov function for the system.

Theorem 5.11 provides an LMI feasibility test to verify robust asymptotic stability against uncertainties that satisfy Assumption 5.9.

It is interesting to compare the numerical complexity of the conditions of Theorem 5.7 with the conditions mentioned in Theorem 5.11. If the uncertainty vector δ is p-dimensional then the vertex set $\boldsymbol{\delta}_g$ has dimension 2^p so that the verification of conditions (5.2.10) requires a feasibility test of

$$2^p + 2^p + p$$

linear matrix inequalities. In this case, also the vertex set $\boldsymbol{\lambda}_g$ has dimension 2^p which implies that the condition of Theorem 5.11 require a feasibility test of

$$2^{2p} + 2^p + p = 4^p + 2^p + p$$

linear matrix inequalities.

Generalizations

By assuming the state evolution map $A(\cdot)$ and the Lyapunov matrix function $K(\cdot)$ to be affine, we were able (Theorem 5.11) to restrict the search of a parameter dependent Lyapunov function for the system (5.2.9) to a finite dimensional subspace. The central idea that led to Theorem 5.11 allows many generalizations to non-affine structures. For example, if $b_1(\delta), b_2(\delta), \ldots, b_p(\delta)$ denote a set of scalar, continuously differentiable basis functions in the uncertain parameter δ, we may assume that $A(\delta)$ and $K(\delta)$ allow for expansions

$$A(\delta) = A_0 + A_1 b_1(\delta) + \cdots + A_p b_p(\delta)$$
$$K(\delta) = K_0 + K_1 b_1(\delta) + \cdots + K_p b_p(\delta).$$

The condition (5.2.10b) in Theorem 5.10 then involves the partial derivative

$$\partial_k K(\delta) = \sum_{j=1}^{p} K_j \partial_k b_j(\delta).$$

The robust stability conditions (5.2.10) in Theorem 5.10 then translate to

$$\sum_{k=1}^{p} K_k b_k(\delta) \succ 0 \quad \text{for all } \delta \in \boldsymbol{\delta} \tag{5.2.13a}$$

$$\sum_{k=1}^{p} \left(\sum_{j=1}^{p} K_j \partial_k b_j(\delta) \lambda_k + [A(\delta)^\top K_j + K_j A(\delta)] b_j(\delta) \right) \prec 0 \quad \text{for all } \delta \in \boldsymbol{\delta} \text{ and } \lambda \in \boldsymbol{\lambda}. \tag{5.2.13b}$$

which is finite dimensional but not yet an LMI feasibility test. Possible basis functions are

(a) **natural basis**
$$b_1(\delta) = \delta_1, \ldots, b_p(\delta) = \delta_p$$
where $\delta_k = \langle e_k, \delta \rangle$ is the kth component of δ in the natural basis $\{e_k\}_{k=1}^{p}$ of $\mathbb{R}^p$.

(b) **multivariate polynomial basis**

$$b_{k_1,\dots,k_p}(\delta) = \delta_1^{k_1}\cdots\delta_p^{k_p}, \quad k_v = 0,1,2,\dots, \quad \text{and } v = 1,\dots,p.$$

Also standard basis functions such as Laguerre polynomials, Zernike functions, Fourier functions, etc. may prove useful in specific applications.

As a conclusion, in this section we derived from the nominal stability characterizations in Chapter 3 the corresponding robust stability tests against both time-invariant and time-varying and rate-bounded parametric uncertainties. We continue this chapter to also generalize the performance characterizations in Chapter 3 to robust performance.

5.3 Robust performance

5.3.1 Robust dissipation

Among the various refinements and generalizations of the notion of a dissipative dynamical system, we mentioned in Section 2.2 the idea of a robust dissipative system. To make this more precise, let $s : W \times Z \to \mathbb{R}$ be a *supply function* associated with the uncertain system S defined in (5.1.1) where the uncertain parameter $\delta(\cdot)$ satisfies Assumption 5.9.

Definition 5.12 (Robust dissipation) The system S defined in (5.1.1) with supply function s is said to be *robustly dissipative* against time-varying uncertainties that satisfy Assumption 5.9 if there exists a function $V : X \times \boldsymbol{\delta} \to \mathbb{R}$ such that the dissipation inequality

$$V(x(t_1),\delta(t_1)) \;\leq\; V(x(t_0),\delta(t_0)) + \int_{t_0}^{t_1} \mathsf{s}(w(t),z(t))\,\mathrm{d}t \tag{5.3.1}$$

holds for all $t_0 \leq t_1$ and all signals (w,x,z,δ) that satisfy (5.1.1) and Assumption 5.9.

Any function V that satisfies (5.3.1) is called a *(parameter dependent) storage function* and (5.3.1) is referred to as the *robust dissipation inequality*. If the composite function $\overline{V}(t) := V(x(t),\delta(t))$ is differentiable as a function of time t, then it is easily seen that (5.3.1) holds for all $t_0 \leq t_1$ and all possible trajectories (w,x,z,δ) if and only if

$$\partial_x V(x,\delta)f(x,w,\delta) + \partial_\delta V(x,\delta)\lambda \;\leq\; \mathsf{s}(w,g(x,w,\delta)) \tag{5.3.2}$$

holds for all points $(x,w,\delta,\lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \boldsymbol{\delta} \times \boldsymbol{\lambda}$. The latter *robust differential dissipation inequality* (5.3.2) makes robust dissipation a *de facto* local property of the functions f, g, the supply function s and the uncertainty sets $\boldsymbol{\delta}$ and $\boldsymbol{\lambda}$.

As in Chapter 2, we will specialize this concept to linear systems with quadratic supply functions and derive explicit tests for the verification of robust dissipation. Suppose that f and g in (5.1.1) are

linear in x and w. This results in the model (5.1.2) in which the uncertainty δ will be time-varying. In addition, suppose that the supply function is a general quadratic form in (w, z) given by (2.3.2) in Chapter 2. A sufficient condition for robust dissipation is then given as follows.

Theorem 5.13 *Suppose that $K : \boldsymbol{\delta} \to \mathbb{S}^n$ is continuously differentiable on a compact set $\boldsymbol{\delta}$ and satisfies*

$$F(K, \delta, \lambda) :=$$
$$\begin{pmatrix} I & 0 \\ A(\delta) & B(\delta) \\ \hline 0 & I \\ C(\delta) & D(\delta) \end{pmatrix}^{\top} \left(\begin{array}{cc|cc} \partial_\delta K(\delta)\lambda & K(\delta) & 0 & 0 \\ K(\delta) & 0 & 0 & 0 \\ \hline 0 & 0 & -Q & -S \\ 0 & 0 & -S^{\top} & -R \end{array} \right) \begin{pmatrix} I & 0 \\ A(\delta) & B(\delta) \\ \hline 0 & I \\ C(\delta) & D(\delta) \end{pmatrix} \preccurlyeq 0 \quad (5.3.3)$$

for all $\delta \in \boldsymbol{\delta}$ and all $\lambda \in \boldsymbol{\lambda}$. Then the uncertain system

$$\begin{cases} \dot{x}(t) = A(\delta(t))x(t) + B(\delta(t))w(t), \quad x(0) = x_0 \\ z(t) = C(\delta(t))x(t) + D(\delta(t))w(t) \end{cases} \quad (5.3.4)$$

where $\delta : \mathbb{R} \to \mathbb{R}^p$ is in the class of continuously differentiable functions satisfying the value and rate constraints stated in Assumption 5.9 is robustly dissipative with respect to the quadratic supply function

$$\mathrm{s}(w, z) = \begin{pmatrix} w \\ z \end{pmatrix}^{\top} \begin{pmatrix} Q & S \\ S^{\top} & R \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix}. \quad (5.3.5)$$

Moreover, in that case, $V(x, \delta) := x^{\top} K(\delta)x$ is a parameter dependent storage function.

Proof. If (5.3.3) holds for all $(\delta, \lambda) \in \boldsymbol{\delta} \times \boldsymbol{\lambda}$, then also $\mathrm{col}(x, w)F(K, \delta, \lambda)\mathrm{col}(x, w) \leq 0$ for all x, w, $\delta \in \boldsymbol{\delta}$ and $\lambda \in \boldsymbol{\lambda}$. But this is precisely the robust differential dissipation inequality with $V(x, \delta) := x^{\top}K(\delta)x$. Hence, (5.3.4) is robustly dissipative with respect to the given quadratic supply function and uncertainty sets that satisfy Assumption 5.9. ∎

Theorem 5.13 provides a sufficient condition for robust dissipation. The condition (5.3.3) is also necessary if the class of storage function $V(x, \delta)$ is restricted to functions that are *quadratic* in x, i.e., functions of the form $V(x, \delta) = x^{\top}K(\delta)x$. A result similar to Theorem 5.13 can be obtained for robust *strictly* dissipative systems.

Definition 5.14 (Robust strict dissipation) The uncertain system S with supply function s is said to be *robust strictly dissipative* against time-varying uncertainties if there exists a storage function $V : X \times \boldsymbol{\delta} \to \mathbb{R}$ and an $\varepsilon > 0$ such that

$$V(x(t_1), \delta(t_1)) \leq V(x(t_0), \delta(t_0)) + \int_{t_0}^{t_1} \mathsf{s}(w(t), z(t))\mathrm{d}t - \varepsilon^2 \int_{t_0}^{t_1} \|w(t)\|^2 \, \mathrm{d}t \quad (5.3.6)$$

for all $t_0 \leq t_1$ and all trajectories (w, x, z, δ) which satisfy (5.1.1).

This notion is a straightforward generalization of Definition 2.5 to uncertain systems. In particular, if the uncertainty parameter δ is time-invariant, then $\boldsymbol{\lambda} = \{0\}$ and $F(K, \delta, \lambda)$ simplifies to an expression that does not depend on λ. Only in this case, one can apply the Kalman-Yakubovich-Popov Lemma 2.11 and infer an equivalent frequency domain characterization. For robust strictly dissipative systems this yields the following statement.

Theorem 5.15 *Consider the uncertain system* (5.1.2) *with time-invariant uncertainties $\delta \in \boldsymbol{\delta}$, $\boldsymbol{\delta}$ compact. The following statements are equivalent.*

(a) *There exists a continuously differentiable function $K : \boldsymbol{\delta} \to \mathbb{S}^n$ such that*

$$
F(K,\delta) := \begin{pmatrix} I & 0 \\ A(\delta) & B(\delta) \\ \hline 0 & I \\ C(\delta) & D(\delta) \end{pmatrix}^\top \left(\begin{array}{cc|cc} 0 & K(\delta) & 0 & 0 \\ K(\delta) & 0 & 0 & 0 \\ \hline 0 & 0 & -Q & -S \\ 0 & 0 & -S^\top & -R \end{array} \right) \begin{pmatrix} I & 0 \\ A(\delta) & B(\delta) \\ \hline 0 & I \\ C(\delta) & D(\delta) \end{pmatrix} \prec 0
\tag{5.3.7}
$$

(b) *For all $(\omega, \delta) \in \mathbb{R} \times \boldsymbol{\delta}$ with $\det(i\omega I - A(\delta)) \neq 0$, the uncertain transfer function $T(s, \delta) := C(\delta)(Is - A(\delta))^{-1}B(\delta) + D(\delta)$ satisfies*

$$
\begin{pmatrix} I \\ T(i\omega, \delta) \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} I \\ T(i\omega, \delta) \end{pmatrix} \succ 0.
\tag{5.3.8}
$$

Moreover, if one of the above equivalent statements holds, then the uncertain system (5.1.2) *is robust strictly dissipative with respect to the supply function* (5.3.5)*.*

5.3.2 Robust quadratic performance

Formulating Theorem 5.15 for robust strictly dissipative systems yields the following generalization of Proposition 3.9 to uncertain dynamical systems.

Theorem 5.16 *Consider the uncertain system* (5.3.4) *where $\delta : \mathbb{R} \to \mathbb{R}^p$ is in the class of continuously differentiable functions satisfying the value and rate constraints in Assumption 5.9. Suppose there exists a continuously differentiable function $K : \boldsymbol{\delta} \to \mathbb{S}^n$ such that*

(a) $K(\delta) \succ 0$ *for all $\delta \in \boldsymbol{\delta}$ and*

(b) *for all $\delta \in \boldsymbol{\delta}$ and $\lambda \in \boldsymbol{\lambda}$*

$$
\begin{pmatrix} I & 0 \\ A(\delta) & B(\delta) \\ \hline 0 & I \\ C(\delta) & D(\delta) \end{pmatrix}^\top \left(\begin{array}{cc|cc} \partial_\delta K(\delta)\lambda & K(\delta) & 0 & 0 \\ K(\delta) & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S \\ 0 & 0 & S^\top & R \end{array} \right) \begin{pmatrix} I & 0 \\ A(\delta) & B(\delta) \\ \hline 0 & I \\ C(\delta) & D(\delta) \end{pmatrix} \prec 0
\tag{5.3.9}
$$

If, in addition, $R \succcurlyeq 0$, then

- *the origin of the uncertain system* (5.3.4) *is exponentially stable and*

- *there exists $\varepsilon > 0$ such that for $x(0) = 0$ and for all $w \in \mathscr{L}_2$ and all uncertain parameter functions $\delta(\cdot)$ satisfying Assumption 5.9 we have*

$$\int_0^\infty \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} dt \leq -\varepsilon^2 \int_0^\infty w^\top(t)w(t)\, dt \qquad (5.3.10)$$

Proof. If $K : \boldsymbol{\delta} \to \mathbb{S}^n$ satisfies the hypothesis then

$$\partial_\delta K(\delta)\lambda + A(\delta)^\top K(\delta) + K(\delta)A(\delta) + C(\delta)^\top RC(\delta) \prec 0.$$

for all $(\delta, \lambda) \in \boldsymbol{\delta} \times \boldsymbol{\lambda}$. With $R \succcurlyeq 0$ it follows that $K(\cdot)$ satisfies (5.2.10) which, by Theorem 5.10, proves the first item. To see the second item, observe that (as in Theorem 5.13) (5.3.9) implies that the system (5.3.4) with the negative of the supply function (5.3.5) is robust strictly dissipative against the given class of uncertainties. Consequently, using (5.3.6) with $V(x, \delta) := x^\top K(\delta)x$, and the fact that $K(\delta) \succ 0$ and $V(0, \delta) = 0$ this gives

$$\int_0^{t_1} \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} dt \ \leq \ -\varepsilon^2 \int_0^{t_1} w^\top(t)w(t)\, dt$$

for all $t_1 \geq 0$ and all trajectories (w, x, z, δ) that are compatible with the system and the uncertainty set. Let $t_1 \to \infty$ to infer (5.3.10) for all $w \in \mathscr{L}_2$ and all δ that satisfy Assumption 5.9. ∎

The inequality (5.3.10) will be referred to as the *robust quadratic performance* property and is the robust variation of the quadratic nominal performance that we introduced in Proposition 3.9. As in Chapter 3 it allows many specializations to define robust passivity, robust H_∞, robust H_2 and a robust L_2 gain performance criterion for uncertain dynamical systems. From this point on, these specializations are straightforward applications of Theorem 5.16 and are left to the reader.

5.4 Further reading

5.5 Exercises

Exercise 1
Consider the system $\dot{x} = A(\delta)x$ with $A(\delta)$ and $\boldsymbol{\delta}$ defined in Example 5.4.

(a) Show that $A(\delta)$ is quadratically stable on the set $0.4\boldsymbol{\delta}$.

(b) Derive a quadratic stability test for the eigenvalues of $A(\delta)$ being located in a disc of radius r with center at the origin.

(c) Can you find the smallest radius r by convex optimization?

Exercise 2

Time-invariant perturbations and arbitrary fast perturbations can be viewed as two extreme cases of time-varying uncertainty sets that satisfy Assumption 5.9. These two extreme manifestations of time-varying perturbations reduce Theorem 5.10 to two special cases.

(a) Show that the result of Theorem 5.7 is obtained as a special case of Theorem 5.11 if $\boldsymbol{\lambda} = \{0\}$.

(b) Show that if $\boldsymbol{\lambda} = [-r, r]^p$ with $r \to \infty$ then the matrices $K_0, \dots, K_p$ satisfying the conditions of Theorem 5.11 necessarily satisfy $K_1 = \dots = K_p = 0$.

Exercise 3

Reconsider the suspension system of Exercise 7 in Chapter 4. Suppose that the road profile $q_0 = 0$ and the active suspension force $f = 0$. Let $\bar{k} = 250$ and $\bar{b} = 50$. The suspension damping is a time-varying uncertain quantity with

$$b_2(t) \in [500 - \bar{b}, 500 + \bar{b}], \quad t \geq 0 \tag{5.5.1}$$

and the suspension stiffness is a time-varying uncertainty parameter with

$$k_2(t) \in [5000 - \bar{k}, 5000 + \bar{k}], \quad t \geq 0. \tag{5.5.2}$$

Let $\delta(t) = \mathrm{col}(b_2(t), k_2(t))$ be the vector containing the uncertain physical parameters.

(a) Let $x = \mathrm{col}(q_1, q_2, \dot{q}_1, \dot{q}_2)$ denote the state of this system and write this system in the form (5.2.1). Verify whether $A(\delta)$ is affine in the uncertainty parameter δ.

(b) Use Proposition 5.3 to verify whether this system is quadratically stable. If so, give a quadratic Lyapunov function for this system.

(c) Calculate vertex matrices $A^1, \dots, A^N$ (superscripts, not powers) such that

$$A(\delta) \in \mathrm{conv}(A^1, \dots, A^N)$$

for all δ satisfying the specifications.

(d) Suppose that b_2 and k_2 are time-varying and that their rates of variation satisfy
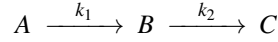
$$|\dot{b}_2| \leq \beta \tag{5.5.3a}$$
$$|\dot{k}_2| \leq \kappa \tag{5.5.3b}$$

where $\beta = 1$ and $\kappa = 2.5$. Use Theorem 5.11 to verify whether there exists a parameter dependent Lyapunov function that proves affine quadratic stability of the uncertain system.

Exercise 4
In Exercise 8 of Chapter 3 we considered the batch chemical reactor where the series reaction

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

takes place. k_1 and k_2 are the kinetic rate constants of the conversions from product A to B and from product B to product C, respectively. We will be interested in the concentration C_B of product B and assume here that the kinetic rate constants are uncertain.

(a) Show that C_B satisfies the differential equation $\ddot{C}_B + (k_1 + k_2)\dot{C}_B + k_1 k_2 C_B = 0$ and represent this system in state space form with state $x = \text{col}(C_A, C_B)$.

(b) Show that the state space system is of the form (5.2.1) where A is an affine function of the kinetic rate constants.

(c) Verify whether this system is quadratically stable in view of jointly uncertain kinetic constants k_1 and k_2 in the range $[.1, 1]$. If so, calculate a Lyapunov function for the uncertain system, if not, try to find a parameter dependent Lyapunov function for this system.

(d) At time $t = 0$ the reactor is injected with an initial concentration $C_{A0} = 10$ (mol/liter) of reactant A while the concentrations $C_B(0) = C_C(0) = 0$. Plot the time evolution of the concentration C_B of reactant B if

$$k_1(t) = 1 - 0.9\exp(-t); \qquad k_2(t) = 0.1 + 0.9\exp(-t).$$

Exercise 5
Let $f : \boldsymbol{\delta} \to \mathbb{R}$ be partially convex and suppose that $\boldsymbol{\delta} = \{\delta \mid \delta_k \in [\underline{\delta}_k, \bar{\delta}_k], k = 1, \ldots, p\}$ with $\underline{\delta}_k \leq \bar{\delta}_k$. Let $\boldsymbol{\delta}_g = \{\delta \mid \delta_k \in \{\underline{\delta}_k, \bar{\delta}_k\}, k = 1, \ldots, p\}$ be the corresponding set of corner points. Show that for all $\gamma \in \mathbb{R}$ we have that
$$f(\delta) \leq \gamma \text{ for all } \delta \in \boldsymbol{\delta}$$
if and only if
$$f(\delta) \leq \gamma \text{ for all } \delta \in \boldsymbol{\delta}_g.$$

Exercise 6
Suppose we are given a system $\dot{x} = Ax + Bw$, $z = Cx + Dw$ with

$$A = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 4 & -3 \\ 1 & -3 & -1 & -3 \\ 0 & 4 & 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(a) Let all coefficients of C deviate from their nominal values by 100%.

   (i) Determine the worst-case H_∞-norm. (Please argue how this can be done!)

   (ii) Compute the worst case H_∞-norm with a common quadratic storage function.

(iii) Can you explain the difference?

(b) Now suppose that only the elements of A deviate from their nominal value by 5%. Let us try to compute a bound on the H_∞ norm with a common quadratic storage function.

   (i) If describing this uncertainty as $A \in \text{conv}\{A_1, \ldots, A_N\}$, how many generators N do you need? Is the direct approach as in the previous exercise expected to be tractable?

   (ii) Determine an LFR: Find a representation of the uncertain system as

$$
\begin{pmatrix} \dot{x} \\ z_1 \\ z \end{pmatrix} = \begin{pmatrix} A_0 & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} x \\ w_1 \\ w \end{pmatrix}, \quad w_1 = \text{diag}(\delta_1 I, \ldots, \delta_{12} I) z_1.
$$

(iii) Consider the LMI problem of minimizing γ such that

$$
X \succ 0, \quad \begin{pmatrix} I & 0 & 0 \\ A_0 & B_1 & B_2 \\ 0 & I & 0 \\ C_1 & D_{11} & D_{12} \\ 0 & 0 & I \\ C_2 & D_{21} & D_{22} \end{pmatrix}^{?} \left(\begin{array}{cc|cc|cc} 0 & X & 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -Q & S & 0 & 0 \\ 0 & 0 & S^T & Q & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array} \right) \begin{pmatrix} I & 0 & 0 \\ A_0 & B_1 & B_2 \\ 0 & I & 0 \\ C_1 & D_{11} & D_{12} \\ 0 & 0 & I \\ C_2 & D_{21} & D_{22} \end{pmatrix} \prec 0
$$

where $Q = \text{diag}(Q_1, \ldots, Q_{12}) \succcurlyeq 0$ and $S = \text{diag}(S_1, \ldots, S_{12})$ is skew-symmetric. Why does this compute a bound on the worst-case H_∞-norm? (Don't forget stability.)

(iv) Compute such an optimal bound for the given system.

Exercise 7 (MIMO circle criterion and extensions)
Consider the nonlinear system

$$\dot{x} = Ax + Bw, \quad z = Cx, \quad w = \Delta(t, z) \tag{5.5.4}$$

where $\Delta : \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^k$ is any Lipschitz continuous function that satisfies, for two matrices K, L, the multi-variable sector condition

$$[\Delta(t, z) - Kz]^T [\Delta(t, z) - Lz] \leq 0 \quad \text{for all} \quad (t, z) \in \mathbb{R} \times \mathbb{R}^l. \tag{5.5.5}$$

(a) Find a symmetric matrix P such that

$$\begin{pmatrix} \Delta(z, t) \\ z \end{pmatrix}^T P \begin{pmatrix} \Delta(z, t) \\ z \end{pmatrix} \geq 0 \quad \text{for all} \quad z, t.$$

(b) Formulate an LMI test for proving global exponential stability of (5.5.4).

(c) With $M(s) = C(sI - A)^{-1}B$ define $G(s) = (I - LM(s))(I - KM(s))^{-1}$. Show that the LMI you found has a solution iff

$$A + BKC \text{ is stable and } G(i\omega)^* + G(i\omega) \succ 0 \text{ for all } \omega \in \mathbb{R}.$$

Is stability of A required for your arguments?

Hint: Apply a congruence transformation with $\begin{pmatrix} I & 0 \\ KC & I \end{pmatrix}$.

(d) Consider now the saturation nonlinearity

$$\Delta(t,z) = \begin{cases} 1 & \text{for } z \geq 1 \\ z & \text{for } |z| \leq 1 \\ -1 & \text{for } z \leq -1 \end{cases} .$$

Show that it satisfies the sector condition (7.8.2) with $K = 0$ and $L = 1$.

(e) Is (5.5.4) with the saturation nonlinearity and

$$C(sI - A)^{-1}B = M(s) = \frac{-4}{(s+1)(\frac{1}{2}s+1)(\frac{1}{3}s+1)}$$

globally exponentially stable?

(f) Let us now suppose the system is excited by an L_2-disturbance d as

$$\dot{x} = Ax + B(w + d), \quad z = Cx, \quad w = \Delta(t,z), \quad e = Cx.$$

Use the multiplier τP with $\tau \geq 0$ to set up an LMI optimization problem which minimizes a guaranteed bound γ on the L_2-gain of the nonlinear system with input d and output e.

(g) Compute such an optimal bound for the saturation nonlinearity in item (d) and the system in item (e).
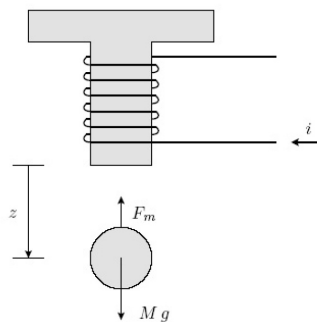
Exercise 8

Figure 5.1: Magnetic levitation system.

Consider the magnetic levitation system as depicted in Figure 5.1. A simple nonlinear model of this system is given by

$$M\frac{d^2}{dt^2}z = Mg - k\frac{i^2}{z^2}, \quad L\frac{d}{dt}i + Ri = v$$

where the voltage v is the input of the system and where the ball position z is the output. The current through the inductor is i; M denotes the mass that is attracted by the coil; L is the magnetic induction of the coil and R its resistance. Suppose that $M = 0.005$, $g = 9.81$, $k = 0.0003$, $L = 0.1$, $R = 2$ (in SI units).

(a) With $v_0 = 2$ Volt, determine the equilibrium (z_0, i_0) of the system with $z_0 > 0$.

(b) Determine a function $f(z, i)$ such that the system can be represented as

$$\frac{d}{dt}\begin{pmatrix} z - z_0 \\ \dot{z} \\ i - i_0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \sqrt{g}f(z,i) & 0 & -\alpha f(z,i) \\ 0 & 0 & -\frac{R}{L} \end{pmatrix}\begin{pmatrix} z - z_0 \\ \dot{z} \\ i - i_0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{L} \end{pmatrix}(v - v_0).$$

This motivates to consider the parameter-dependent system

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ \sqrt{g}\delta(t) & 0 & -\alpha\delta(t) \\ 0 & 0 & -\frac{R}{L} \end{pmatrix}x + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{L} \end{pmatrix}u, \quad z = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}x \qquad (5.5.6)$$

where $\delta(t)$ varies in the interval defined by a 50%-deviation from the nominal (mid-point) value $\delta_0 = f(z_0, i_0)$. Let us denote the LTI system that results from (5.5.6) for $\delta(t) = \delta_0$ by the transfer function $G(s)$. The goal is to design a stabilizing dynamic control law $u = K(z - r)$ such that the output z tracks the reference signal r.

(c) Use an H_∞-synthesis procedure to design such a controller for the generalized plant

$$e = Gu - r, \quad y = Gu - r$$

To do this, incorporate weightings

$$W_e(s) = \frac{2s + 100}{5s + 1} \quad \text{and} \quad W_u(s) = 8 * 10^{-5}$$

on the tracking error e and the control input u. Analyze the resulting controlled system by providing relevant plots of frequency and step responses. Draw conclusions about the quality of the design.

(d) For the designed H_∞-controller, compute the quadratic stability margin of the controlled system. What do you conclude about the allowable values of $f(z(t), i(t))$ along a trajectory without endangering stability?

(e) Apply a reference signal $r(t) = a\sin(\omega t)$ with various amplitudes a and frequencies ω to the system. Confirm the previous stability guarantees by making a nonlinear simulation, and investigate the possibility to destabilize the controlled system.

Chapter 6

Uncertainty and linear fractional representations

Control systems need to operate well in the face of many types of uncertainties. That is, stability and performance of controlled systems need to be as robust as possible against perturbations, model uncertainties and un-modeled dynamics. These uncertainties may be known, partly known or completely unknown. It will be one of the fundamental insights in this chapter that large classes of uncertainties can be modeled in a unified manner by representing uncertain systems as interconnections of nominal systems and uncertain components. This insight is at the basis of the notion of linear fractional representations and proves instrumental to derive guarantees for robust stability and robust performance against linear time invariant uncertainties of complex systems. A main advantage of this notion is that complex and diverse robustness questions in control can be transformed to a specific algebraic problem and subsequently be solved. It is of quite some independent interest to understand the emergence of robustness in algebraic problems and to convey the essential techniques how to handle them.

6.1 Linear fractional representations of rational functions

To set the stage, consider the uncertain autonomous system

$$\dot{x} = F(\delta)x \tag{6.1.1}$$

where $x(t) \in \mathbb{R}^n$ is an n dimensional state vector, and $F(\delta)$ is a real matrix-valued function which depends on a p-dimensional parameter vector $\delta = \mathrm{col}(\delta_1, \ldots, \delta_p)$ that is assumed to belong to a parameter set $\boldsymbol{\delta} \subset \mathbb{R}^p$. Apart from a change of notation, this is the same system as (5.2.1) that we studied in Chapter 5.
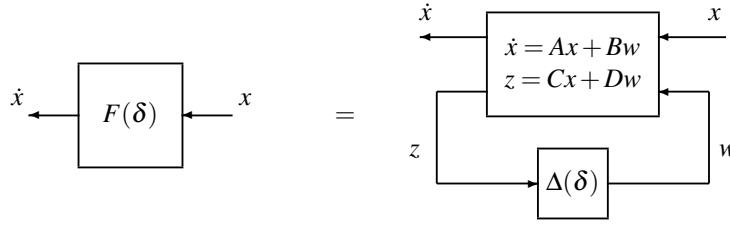
Figure 6.1: Linear fractional representation of (6.1.1)

The aim of this section is to represent the uncertain system (6.1.1) in the form

$$\begin{pmatrix} \dot{x} \\ z \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}, \qquad w = \Delta(\delta)z \qquad (6.1.2)$$

where Δ is a function that depends *linearly* on δ. This standard form will be called a *linear fractional representation* and the general idea is depicted in Figure 6.1.

In this chapter it will be demonstrated that a linear fractional representation is, in fact, a very general and powerful tool to represent uncertainty in dynamical systems. Among the key results, we will show that any system of the form (6.1.1) where F depends *rationally* on δ can be written in the form (6.1.2). To study these representations, it will be essential to identify, as usual, the *matrix* $F(\delta)$ with the *linear function* $F(\delta): \mathbb{R}^n \to \mathbb{R}^n$ that maps $\eta \mapsto \xi := F(\delta)\eta$.

Definition 6.1 A *linear fractional representation* (LFR) of $F(\delta)$ is a pair $(H, \Delta(\delta))$ where

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is a constant partitioned real-valued matrix and Δ is a *linear* function of δ such that for all δ for which $I - D\Delta(\delta)$ is invertible and for all (η, ξ) there holds $\xi = F(\delta)\eta$ if and only if there exist vectors w and z such that

$$\begin{pmatrix} \xi \\ z \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \eta \\ w \end{pmatrix}, \qquad w = \Delta(\delta)z. \qquad (6.1.3)$$

The LFR is said to be *well posed* at δ if $I - D\Delta(\delta)$ is non-singular. We occasionally call $\Delta(\delta)$ the *parameter-block* of the LFR.

               Compilation: January 2015

Hence, if (6.1.3) is a well posed LFR of $F(\delta)$ at δ then

$$\xi = F(\delta)\eta \quad\Longleftrightarrow\quad \exists\, w, z: \quad \begin{cases} \xi = A\eta + Bw \\ z = C\eta + Dw \\ w = \Delta(\delta)z \end{cases}$$

$$\Longleftrightarrow\quad \exists\, z: \quad \begin{cases} (I - D\Delta(\delta))z = C\eta \\ \xi = B\Delta(\delta)z + A\eta \end{cases}$$

$$\Longleftrightarrow\quad \exists\, z: \quad \begin{cases} z = (I - D\Delta(\delta))^{-1}C\eta \\ \xi = B\Delta(\delta)z + A\eta \end{cases}$$

$$\Longleftrightarrow\quad \xi = [A + B\Delta(\delta)(I - D\Delta(\delta))^{-1}C]\eta$$

In other words, in that case, the equations (6.1.3) define a linear mapping and

$$F(\delta) = A + B\Delta(\delta)(I - D\Delta(\delta))^{-1}C. \tag{6.1.4}$$

Whenever well posed, we will write this mapping as $F(\delta) = H \star \Delta(\delta)$, the *star product* of H with $\Delta(\delta)$. Let us agree that, from this point on, the usage of the expression $F(\delta) = H \star \Delta(\delta)$ will implicitly mean that $I - D\Delta(\delta)$ is invertible.

Since $\Delta(\delta)$ is linear in δ, the expression (6.1.4) reveals that any $F(\delta)$ that admits an LFR $(H, \Delta(\delta))$ must be a *rational function* of δ. Moreover, since $F(0) = H \star \Delta(0) = A$, it is clear that zero is not a singular point (a pole) of F. The main point of this section is to prove that the converse is true as well. That is, we will show that *any matrix-valued multi-variable rational function without singular points in the origin admits an LFR*.

Before proving this essential insight we first derive some elementary operations that allow manipulations with LFR's. This will prove useful especially in later chapters where all operations on LFR's are fully analogous to manipulating (feedback) interconnections of linear systems and their state-space realizations. Let us summarize some of the most important operations for a single LFR $F(\delta) = H \star \Delta(\delta)$ or for two LFR's $F_1(\delta) = H_1 \star \Delta_1(\delta)$ and $F_2(\delta) = H_2 \star \Delta_2(\delta)$ assuming fully compatible matrix dimensions.

Summation
As depicted in Figure 6.2, the sum $\xi = [F_1(\delta) + F_2(\delta)]\eta$ admits the LFR

$$\begin{pmatrix} \xi \\ \hline z_1 \\ z_2 \end{pmatrix} = \left(\begin{array}{c|cc} A_1 + A_2 & B_1 & B_2 \\ \hline C_1 & D_1 & 0 \\ C_2 & 0 & D_2 \end{array} \right) \begin{pmatrix} \eta \\ \hline w_1 \\ w_2 \end{pmatrix}, \qquad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \Delta_1(\delta) & 0 \\ 0 & \Delta_2(\delta) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$
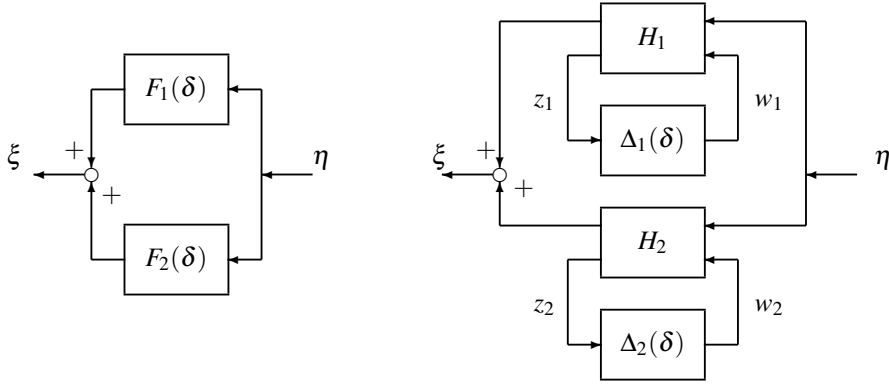
Multiplication
As shown in Figure 6.3, the product $\xi = F_1(\delta)F_2(\delta)\eta$ admits the LFR

$$\begin{pmatrix} \xi \\ \hline z_1 \\ z_2 \end{pmatrix} = \left(\begin{array}{c|cc} A_1A_2 & B_1 & A_1B_2 \\ \hline C_1A_2 & D_1 & C_1B_2 \\ C_2 & 0 & D_2 \end{array} \right) \begin{pmatrix} \eta \\ \hline w_1 \\ w_2 \end{pmatrix}, \qquad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \Delta_1(\delta) & 0 \\ 0 & \Delta_2(\delta) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$
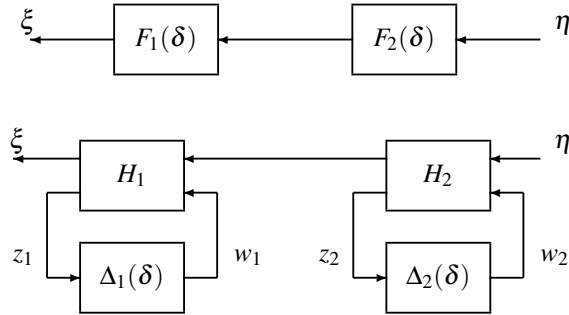
Figure 6.2: Sum of LFR's is LFR

Figure 6.3: Product of LFR's if LFR

Augmentation

For arbitrary matrices L and R, the augmented system $\xi = LF(\delta)R\eta$ admits the LFR

$$\begin{pmatrix} \xi \\ z \end{pmatrix} = \begin{pmatrix} LAR & LB \\ CR & D \end{pmatrix} \begin{pmatrix} \eta \\ w \end{pmatrix}, \qquad w = \Delta(\delta)z.$$

This simple result allows to construct LFR's of row, column and diagonal augmentations by summation. For example, by using that

$$\begin{pmatrix} F_1(\delta) & F_2(\delta) \end{pmatrix} = F_1(\delta)\begin{pmatrix} I & 0 \end{pmatrix} + F_2(\delta)\begin{pmatrix} 0 & I \end{pmatrix}$$

$$\begin{pmatrix} F_1(\delta) \\ F_2(\delta) \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix} F_1(\delta) + \begin{pmatrix} 0 \\ I \end{pmatrix} F_2(\delta)$$

$$\begin{pmatrix} F_1(\delta) & 0 \\ 0 & F_2(\delta) \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix} F_1(\delta)\begin{pmatrix} I & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} F_2(\delta)\begin{pmatrix} 0 & I \end{pmatrix}.$$
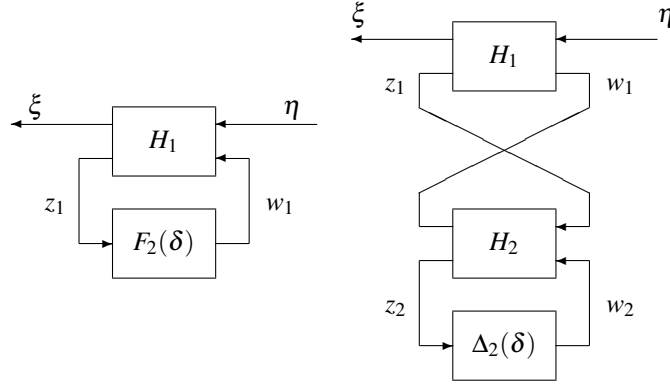
Figure 6.4: LFR of LFR is LFR

Inversion
If A is invertible then $\eta = F(\delta)^{-1}\xi$ admits the LFR

$$\begin{pmatrix} \eta \\ z \end{pmatrix} = \begin{pmatrix} A^{-1} & -A^{-1}B \\ CA^{-1} & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} \xi \\ w \end{pmatrix}, \qquad w = \Delta(\delta)z.$$

LFR of LFR is LFR
As illustrated in Figure 6.4, if $I - A_2 D_1$ is invertible then $\xi = [H_1 \star F_2(\delta)]\eta$ can be written as

$$H_1 \star F_2(\delta) = H_1 \star (H_2 \star \Delta_2(\delta)) = H_{12} \star \Delta_2(\delta)$$

where H_{12} is defined by

$$\begin{pmatrix} \xi \\ z_2 \end{pmatrix} = \underbrace{\begin{pmatrix} A_1 + B_1(I - A_2 D_1)^{-1} A_2 C_1 & B_1(I - A_2 D_1)^{-1} B_2 \\ C_2 C_1 + C_2 D_1(I - A_2 D_1)^{-1} A_2 C_1 & D_2 + C_2 D_1(I - A_2 D_1)^{-1} B_2 \end{pmatrix}}_{H_{12}} \begin{pmatrix} \eta \\ w_2 \end{pmatrix}$$

$$w_2 = \Delta_2(\delta)z_2.$$

Hence, an LFR of an LFR is an LFR. This expression is particularly relevant for the shifting and scaling of parameters. Indeed, with $D_2 = 0$ the LFR

$$F_2(\delta) = H_2 \star \Delta_2(\delta) = A_2 + B_2 \Delta_2(\delta) C_2$$

represents a joint scaling (by B_2 and C_2) and shifting (by A_2) of the uncertainty block $\Delta_2(\delta)$. If the parameter block $\Delta_2(\delta)$ is not shifted, i.e., if $A_2 = 0$ this leads to the even more special case:

$$H_1 \star [B_2 \Delta_2(\delta) C_2] = \tilde{H}_{12} \star \Delta_2(\delta) \text{ with } \tilde{H}_{12} = \begin{pmatrix} A_1 & B_1 B_2 \\ C_2 C_1 & C_2 D_1 B_2 \end{pmatrix}. \tag{6.1.5}$$

Also, if $\Delta_2(\delta)$ is of smaller dimension than $B_2 \Delta_2(\delta) C_2$, the latter formula reveals that one can compress $H_1 \star [B_2 \Delta_2(\delta) C_2]$ to the more compact LFR $\tilde{H}_{12} \star \Delta_2(\delta)$.

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Order reduction of LFR's

In addition, from (6.1.5) it follows that any LFR with a general linear parameter-block $\Delta(\delta)$ can be transformed into an equivalent one with the block-diagonal parameter dependence

$$\Delta_d(\delta) = \begin{pmatrix} \delta_1 I_{d_1} & & 0 \\ & \ddots & \\ 0 & & \delta_p I_{d_p} \end{pmatrix} \qquad (6.1.6)$$

where $I_{d_1}, \ldots, I_{d_p}$ are identity matrices of sizes $d_1, \ldots, d_p$ respectively. In this case we call the positive integer vector $d = (d_1, \ldots, d_p)$ the *order* of the LFR. Indeed, if $\Delta(\delta)$ depends linearly on δ, there exist suitable coefficient matrices Δ_j such that

$$w = \Delta(\delta)z = [\delta_1\Delta_1 + \cdots + \delta_p\Delta_p]z. \qquad (6.1.7)$$

View the matrices $\delta_j I$ as non-dynamic system components and rename the input and output signals of each of these subsystems. Then $w = \Delta(\delta)z$ is equivalent to $w = w_1 + \cdots + w_p$ with $w_j = \delta_j z_j = [\delta_j I_j]z_j$, $z_j = \Delta_j z$ which is the desired alternative LFR. We observe that the size of I_j, which determines how often δ_j has to be repeated, corresponds to the dimension of the signals w_j and z_j. Since the coefficient matrices Δ_j have often small rank in practice, this procedure can be adapted to reduce the order of the LFR. One just needs to perform the factorizations

$$\Delta_1 = L_1 R_1, \quad \ldots \quad , \Delta_p = L_p R_p$$

such that the number of columns and rows of L_j and R_j equal the rank of Δ_j and are, therefore, as small as possible. These full-rank factorizations can be easily computed by a Gauss-elimination algorithm or by applying singular value decompositions. Then (6.1.7) reads as

$$w = \left[L_1(\delta_1 I_{d_1})R_1 + \cdots + L_p(\delta_p I_{d_p})R_p\right] z$$

and leads to the following smaller order LFR of $w = \Delta(\delta)z$:

$$\begin{pmatrix} w \\ z_1 \\ \vdots \\ z_p \end{pmatrix} = \left(\begin{array}{c|ccc} 0 & L_1 & \cdots & L_p \\ \hline R_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_p & 0 & \cdots & 0 \end{array}\right) \begin{pmatrix} z \\ w_1 \\ \vdots \\ w_p \end{pmatrix}, \quad \begin{pmatrix} w_1 \\ \vdots \\ w_p \end{pmatrix} = \begin{pmatrix} \delta_1 I_{d_1} & & 0 \\ & \ddots & \\ 0 & & \delta_p I_{d_p} \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_p \end{pmatrix}.$$

Since an LFR of an LFR is an LFR, it is straightforward to combine this LFR of $w = \Delta(\delta)z$ with that of $\xi = F(\delta)\eta$ to obtain the following new LFR of $\xi = F(\delta)\eta$ with a parameter-block that admits the desired diagonal structure:

$$\begin{pmatrix} \xi \\ z_1 \\ \vdots \\ z_p \end{pmatrix} = \left(\begin{array}{c|ccc} A & BL_1 & \cdots & BL_p \\ \hline R_1 C & R_1 DL_1 & \cdots & R_1 DL_p \\ \vdots & \vdots & \ddots & \vdots \\ R_p C & R_p DL_1 & \cdots & R_p DL_p \end{array}\right) \begin{pmatrix} \eta \\ w_1 \\ \vdots \\ w_p \end{pmatrix}, \quad w_j = [\delta_j I_{d_j}]z_j.$$

6.2 Construction of linear fractional representations

Let us now turn to the construction of linear fractional representations for rational functions $F(\delta)$. We first consider the simplest case in which $F(\delta)$ depends affinely on δ. Then $F(\delta) = F_0 + \Delta(\delta)$ with $\Delta(\delta)$ a linear map. An LFR of $\xi = F(\delta)\eta$ is then given by

$$\begin{pmatrix} \xi \\ z \end{pmatrix} = \begin{pmatrix} F_0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \eta \\ w \end{pmatrix}, \quad w = \Delta(\delta)z.$$

As just described, one can polish this representation further towards an LFR with a block-diagonal parameter-block.

We proceed with a slightly more involved case where $F(\delta)$ is an uncertain matrix which has polynomial dependence on a single variable δ. That is,

$$\xi = [F_0 + \delta F_1 + \cdots + \delta^p F_p]\eta. \tag{6.2.1}$$

To achieve an efficient LFR, we rewrite this as

$$\xi = F_0\eta + \delta[F_1 + \delta[F_2 + \cdots + \delta[F_{p-1} + \delta F_p]\cdots]]\eta. \tag{6.2.2}$$

Then we can separate the uncertainties as

$$\xi = F_0\eta + w_1, \quad w_1 = \delta z_1, \quad z_1 = F_1\eta + w_2, \quad w_2 = \delta z_2, \quad z_2 = F_2\eta + w_3, \ldots$$
$$\ldots w_{p-1} = \delta z_{p-1}, \quad z_{p-1} = F_{p-1}\eta + w_p, \quad w_p = \delta z_p, \quad z_p = F_p\eta.$$

With $w = \mathrm{col}(w_1, \ldots, w_p)$, $z = \mathrm{col}(z_1, \ldots, z_p)$, an LFR of the matrix polynomial (6.2.1) then reads as

$$\begin{pmatrix} \xi \\ \hline z_1 \\ \vdots \\ z_{p-1} \\ z_p \end{pmatrix} = \left(\begin{array}{c|cccc} F_0 & I & \cdots & 0 & 0 \\ \hline F_1 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ F_{p-1} & 0 & \cdots & 0 & I \\ F_p & 0 & \cdots & 0 & 0 \end{array} \right) \begin{pmatrix} \eta \\ \hline w_1 \\ \vdots \\ w_{p-1} \\ w_p \end{pmatrix}, \quad w = \delta z.$$

Next, assume that $F(\delta)$ is a single-parameter matrix-valued rational function without pole in zero. Then it is straightforward to construct a square polynomial matrix D such that $DG = N$ is itself polynomial. For example, one can determine the common multiple of the denominators in each row of $F(\delta)$ and collect these scalar polynomials on the diagonal of D. Since $F(\delta)$ has no pole in zero, none of the element denominators of F vanish in zero. Hence, this procedure will result in a D matrix for which $D(0)$ will be nonsingular. Therefore, $\det(D(\delta))$ does not vanish identically zero which implies that $F(\delta) = D(\delta)^{-1}N(\delta)$ for all δ for which $D(\delta)$ is non-singular. Thus, one can construct polynomial matrices

$$D(\delta) = D_0 + \delta D_1 + \cdots + \delta^p D_p$$
$$N(\delta) = N_0 + \delta N_1 + \cdots + \delta^p N_p$$

with either $D_p \neq 0$ or $N_p \neq 0$ and with the following properties:

- D is square and $D(0) = D_0$ is non-singular.

- If δ is chosen such that $D(\delta)$ is invertible then F has no pole in δ, and $\xi = F(\delta)\eta$ is equivalent to $D(\delta)\xi = N(\delta)\eta$.

Now, the expression $D(\delta)\xi = N(\delta)\eta$ reads as

$$[D_0 + \delta D_1 + \cdots + \delta^p D_p]\,\xi = [N_0 + \delta N_1 + \cdots + \delta^p N_p]\,\eta$$

which is equivalent to

$$\xi = D_0^{-1}N_0\eta + \delta(D_0^{-1}N_1\eta - D_0^{-1}D_1\xi) + \cdots + \delta^p(D_0^{-1}N_p\eta - D_0^{-1}D_p\xi)$$

and hence to

$$\xi = D_0^{-1}[N_0\eta + w_1],\ w_1 = \delta z_1,\ z_1 = \sum_{j=1}^{p}\delta^{j-1}[N_j\eta - D_jD_0^{-1}N_0\eta - D_jD_0^{-1}w_1].$$

Exactly as described for polynomial dependence this leads to the LFR

$$\left(\begin{array}{c} \xi \\ \hline z_1 \\ \vdots \\ z_{p-1} \\ z_p \end{array}\right) = \left(\begin{array}{c|cccc} D_0^{-1}N_0 & D_0^{-1} & \cdots & 0 & 0 \\ \hline N_1 - D_1D_0^{-1}N_0 & -D_1D_0^{-1} & I & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ N_{p-1} - D_{p-1}D_0^{-1}N_0 & -D_{p-1}D_0^{-1} & \cdots & 0 & I \\ N_p - D_pD_0^{-1}N_0 & -D_pD_0^{-1} & \cdots & 0 & 0 \end{array}\right) \left(\begin{array}{c} \eta \\ \hline w_1 \\ \vdots \\ w_{p-1} \\ w_p \end{array}\right), \qquad (6.2.3)$$

$$w_1 = \delta z_1,\ \ldots,\ w_p = \delta z_p. \qquad (6.2.4)$$

If this LFR is well-posed at δ, then $\xi = 0$ implies $\eta = 0$, which in turn shows that $D(\delta)$ is invertible. Therefore we can conclude that (6.2.3) is indeed an LFR of $\xi = F(\delta)\eta$ as desired.

Remark 6.2 We have essentially recapitulated how to construct a realization for the rational function $G(s) := F(1/s)$ in the variable s. Since F has no pole in zero, $G(s)$ is proper. There exists a large body of literature how to compute efficient minimal realizations of G, either by starting with polynomial fractional representations that are coprime, or by reducing (6.2.3) with state-space techniques. By constructing a minimal realization $G(s) = A + B(Is - D)^{-1}C$, (i.e. a realization in which the matrix D has minimal dimension) then

$$F(\delta) = A + B(\frac{1}{\delta}I - D)^{-1}C = A + B\delta(I - D\delta)^{-1}C = \left(\begin{array}{cc} A & B \\ C & D \end{array}\right) \star \delta I_n$$

with $n = \dim(A)$. This leads to the minimal sized LFR

$$\left(\begin{array}{c} \xi \\ z \end{array}\right) = \left(\begin{array}{cc} A & B \\ C & D \end{array}\right)\left(\begin{array}{c} \eta \\ w \end{array}\right), \quad w = \delta z$$

of $F(\delta)$ with rational one-parameter dependence or, equivalently, to an LFR representation

$$\left(\begin{array}{c} \xi \\ z \end{array}\right) = \left(\begin{array}{cc} A & B \\ C & D \end{array}\right)\left(\begin{array}{c} \eta \\ w \end{array}\right), \quad w = \frac{1}{s}z$$

of the transfer function $G(s)$ that maps input η to output ξ.

Remark 6.3 Let us choose an arbitrary mapping $\pi : \{1,\ldots,p\} \to \{1,\ldots,p\}$. If replacing $w_j = [\delta I]z_j$ in (6.2.3) with $w_j = [\delta_{\pi(j)}I]z_j$, we can follow the above derivation backwards to observe that one obtains an LFR of

$$[D_0 + \delta_{\pi(1)}D_1 + (\delta_{\pi(1)}\delta_{\pi(2)})D_2 + \cdots + (\delta_{\pi(1)}\cdots\delta_{\pi(p)})D_p]\xi =$$
$$= [N_0 + \delta_{\pi(1)}N_1 + (\delta_{\pi(1)}\delta_{\pi(2)})N_2 + \cdots + (\delta_{\pi(1)}\cdots\delta_{\pi(p)})N_p]\eta.$$

with a specific multi-linear dependence of $D(\delta)$ and $N(\delta)$ on δ.

Let us finally consider a general multi-variable matrix-valued rational function $F(\delta)$ without pole in zero. This means that each of its elements can be represented as

$$\frac{\displaystyle\sum_{j_1=0}^{m_1}\cdots\sum_{j_p=0}^{m_p}\alpha_{j_1,\ldots,j_p}\,\delta_1^{j_1}\cdots\delta_p^{j_p}}{\displaystyle\sum_{j_1=0}^{l_1}\cdots\sum_{j_p=0}^{l_p}\beta_{j_1,\ldots,j_p}\,\delta_1^{j_1}\cdots\delta_p^{j_p}} \quad \text{with} \ \ \beta_{0,\ldots,0} \neq 0.$$

Literally following the construction as described for one-variable polynomials, we now construct multi-variable polynomial matrices D and N with $DF = N$ and nonsingular $D(0)$. Likewise, for all δ for which $D(\delta)$ is non-singular one concludes that $\xi = F(\delta)\eta$ is equivalent to $D(\delta)\xi = N(\delta)\eta$ or to the kernel representation

$$0 = H(\delta)\begin{pmatrix} \eta \\ \xi \end{pmatrix} \quad \text{with} \ \ H(\delta) = \begin{pmatrix} N(\delta) & -D(\delta) \end{pmatrix}. \tag{6.2.5}$$

Decompose $H(\delta)$ according to

$$H(\delta) = H_0 + \delta_1 H_1^1(\delta) + \cdots + \delta_p H_p^1(\delta)$$

where H_j^1 are polynomial matrices whose degrees are strictly smaller than those of H. Then (6.2.5) is equivalent to

$$0 = H_0\begin{pmatrix} \eta \\ \xi \end{pmatrix} + \sum_{j=1}^{p} w_j^1, \quad w_j^1 = \delta_j z_j^1, \ \ z_j^1 = H_j^1(\delta)\begin{pmatrix} \eta \\ \xi \end{pmatrix}. \tag{6.2.6}$$

Let us stress that in this decomposition the channel j is considered absent (empty) whenever H_j^1 is the zero polynomial matrix. With $w^1 = \text{col}(w_1^1,\ldots,w_p^1)$, $z^1 = \text{col}(z_1^1,\ldots,z_p^1)$, $E_0 := (I \ \cdots \ I)$, $H^1 = \text{col}(H_1^1,\ldots,H_p^1)$, the expression (6.2.6) is more compactly written as

$$0 = H_0\begin{pmatrix} \eta \\ \xi \end{pmatrix} + E_0 w^1, \quad z^1 = H^1(\delta)\begin{pmatrix} \eta \\ \xi \end{pmatrix}, \ \ w^1 = \text{diag}(\delta_1 I,\ldots,\delta_p I)z^1. \tag{6.2.7}$$

We are now in the position to iterate. For this purpose, decompose

$$H^1(\delta) = H_1 + \delta_1 H_1^2(\delta) + \cdots + \delta_p H_p^2(\delta)$$

and pull parameters out of the middle relation in (6.2.7) according to

$$z_1 = H_1 \begin{pmatrix} \eta \\ \xi \end{pmatrix} + \sum_{j=1}^{p} w_j^2, \quad w_j^2 = \delta_j z_j^2, \quad z_j^2 = H_j^2(\delta) \begin{pmatrix} \eta \\ \xi \end{pmatrix}.$$

Again using compact notation, (6.2.7) is then equivalent to

$$0 = H_0 \begin{pmatrix} \eta \\ \xi \end{pmatrix} + E_0 w^1, \quad z^1 = H_1 \begin{pmatrix} \eta \\ \xi \end{pmatrix} + E_1 w^2, \quad z^2 = H^2(\delta) \begin{pmatrix} \eta \\ \xi \end{pmatrix},$$
$$w^1 = \text{diag}(\delta_1 I, \ldots, \delta_p I) z^1, \quad w^2 = \text{diag}(\delta_1 I, \ldots, \delta_p I) z^2.$$

Continue in this fashion. Then at each iteration the degree of H^j is strictly decreased, and the iteration stops after k steps if $H^k(\delta) = H_k$ is just a constant matrix. In this last step one arrives at

$$0 = H_0 \begin{pmatrix} \eta \\ \xi \end{pmatrix} + E_0 w^1, \quad z^j = H_j \begin{pmatrix} \eta \\ \xi \end{pmatrix} + E_j w^{j+1}, \quad w^j = \text{diag}(\delta_1 I, \ldots, \delta_p I) z^j, \quad j = 1, \ldots, k$$

with E_k and w^{k+1} being empty. Now turn this implicit relation into a genuine input-output relation. For this purpose, partition

$$H_j = \begin{pmatrix} N_j & -D_j \end{pmatrix}$$

conformably with $H = (N - D)$. Recall that $D_0 = D(0)$ is non-singular and hence

$$0 = H_0 \begin{pmatrix} \eta \\ \xi \end{pmatrix} + E_0 w^1 \text{ is equivalent to } \xi = D_0^{-1} N_0 \eta + D_0^{-1} E_0 w^1.$$

After performing this substitution we arrive at the desired explicit LFR of the input-output relation $\xi = F(\delta)\eta$. It is given by

$$\xi = D_0^{-1} N_0 \eta + D_0^{-1} E_0 w^1, \quad z^j = [N_j - D_j D_0^{-1} N_0]\eta - [D_j D_0^{-1} E_0]w^1 + E_j w^{j+1},$$
$$w^j = \text{diag}(\delta_1 I, \ldots, \delta_p I) z^j, \quad j = 1, \ldots, k.$$

In matrix form, this reads

$$\begin{pmatrix} \frac{\xi}{z^1} \\ \vdots \\ z^{k-1} \\ z^k \end{pmatrix} = \left(\begin{array}{c|cccc} D_0^{-1} N_0 & D_0^{-1} E_0 & \cdots & 0 & 0 \\ \hline N_1 - D_1 D_0^{-1} N_0 & -D_1 D_0^{-1} E_0 & E_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ N_{k-1} - D_{k-1} D_0^{-1} N_0 & -D_{k-1} D_0^{-1} E_0 & \cdots & 0 & E_{k-1} \\ N_k - D_k D_0^{-1} N_0 & -D_k D_0^{-1} E_0 & \cdots & 0 & 0 \end{array} \right) \begin{pmatrix} \frac{\eta}{w^1} \\ \vdots \\ w^{k-1} \\ w^k \end{pmatrix},$$
$$w^1 = \text{diag}(\delta_1 I, \ldots, \delta_p I) z^1$$
$$\vdots$$
$$w^k = \text{diag}(\delta_1 I, \ldots, \delta_p I) z^k.$$

Of course this LFR can again be modified to one with the diagonal parameter-block (6.1.6), which just amounts to reordering variables in the present situation. We have constructively proved the

desired representation theorem for matrices $F(\delta)$ that depend rationally on the parameter vector $\delta \in \mathbb{R}^p$. The presented technique is suited to lead to reasonably sized LFR's in practice. We remark that the steps that we performed in the construction of LFR's of rational functions $F(\delta)$ can be largely automated.

Theorem 6.4 *Suppose that all entries of the matrix $F(\delta)$ are rational functions of $\delta = (\delta_1, \ldots, \delta_p)$ whose denominators do not vanish at $\delta = 0$. Then there exist matrices A, B, C, D and non-negative integers $d_1, \ldots, d_p$ such that, for any pair (η, ξ) and with $\Delta(\delta) := \mathrm{diag}(\delta_1 I_{d_1}, \ldots, \delta_p I_{d_p})$,*

$$\xi = F(\delta)\eta$$

if and only if there exists w, z such that

$$\begin{pmatrix} \xi \\ z \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \eta \\ w \end{pmatrix}, \quad w = \Delta(\delta)z$$

for all δ with $\det(I - D\Delta(\delta)) \neq 0$.

Example 6.5 As a simple example of the above construction, consider the matrix

$$F(\delta) := \begin{pmatrix} -1 & 2\delta_1 \\ \frac{-1}{1+\delta_1} & -4+3\delta_2 \end{pmatrix}$$

Applying the previous construction, we first observe that $\xi = F(\delta)\eta$ can be written as

$$\begin{pmatrix} 1 & 0 \\ 0 & 1+\delta_1 \end{pmatrix} \xi = \begin{pmatrix} -1 & 2\delta_1 \\ -1 & (-4+3\delta_2)(1+\delta_1) \end{pmatrix} \eta$$

which is polynomial in the parameter vector. This gives in the first step

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xi + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w_1 = \begin{pmatrix} -1 & 0 \\ -1 & -4+3\delta_2 \end{pmatrix} \eta + \begin{pmatrix} 2 \\ -4+3\delta_2 \end{pmatrix} w_2,$$

$$w_1 = \delta_1 z_1, \quad z_1 = \xi_2, \quad w_2 = \delta_1 z_2, \quad z_2 = \eta_2$$

and in the second and last step

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xi + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w_1 = \begin{pmatrix} -1 & 0 \\ -1 & -4 \end{pmatrix} \eta + \begin{pmatrix} 2 \\ -4 \end{pmatrix} w_2 + \begin{pmatrix} 0 \\ 3 \end{pmatrix} w_3,$$

$$w_1 = \delta_1 z_1, \quad z_1 = \xi_2, \quad w_2 = \delta_1 z_2, \quad z_2 = \eta_2, \quad w_3 = \delta_2 z_3, \quad z_3 = \eta_2 + w_2.$$

It follows that $F(\delta) = H \star \Delta(\delta)$ with

$$H = \left(\begin{array}{cc|ccc} -1 & 0 & 0 & 2 & 0 \\ -1 & -4 & -1 & -4 & 3 \\ \hline -1 & -4 & -1 & -4 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array} \right), \quad \Delta(\delta) = \begin{pmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_1 & 0 \\ 0 & 0 & \delta_2 \end{pmatrix}$$

where the parameter block is in diagonal form. Note that the order of this LFR is $d = (2,1)$.

Example 6.6 To illustrate the construction on a more elaborate example, consider

$$F(\delta) := \frac{1}{(\delta_2 - 1)(\delta_2 + \delta_1^2 + \delta_1 - 1)} \begin{pmatrix} 2\delta_2^2 - \delta_2 + 2\delta_1^2\delta_2 - \delta_1^2 + 2\delta_1 & \delta_1\delta_2 + \delta_1^2 + \delta_2 + \delta_1 \\ (-2\delta_1\delta_2 + 3\delta_1 + 2\delta_2 - 1)\delta_1 & (\delta_1\delta_2 + 1)\delta_1 \end{pmatrix}$$

together with the equation

$$\xi = F(\delta)\eta.$$

Then

$$\underbrace{\begin{pmatrix} (\delta_2 - 1)(\delta_2 + \delta_1^2 + \delta_1 - 1) & 0 \\ 0 & (\delta_2 - 1)(\delta_2 + \delta_1^2 + \delta_1 - 1) \end{pmatrix}}_{D(\delta)} \xi =$$

$$= \underbrace{\begin{pmatrix} 2\delta_2^2 - \delta_2 + 2\delta_1^2\delta_2 - \delta_1^2 + 2\delta_1 & \delta_1\delta_2 + \delta_1^2 + \delta_2 + \delta_1 \\ (-2\delta_1\delta_2 + 3\delta_1 + 2\delta_2 - 1)\delta_1 & (\delta_1\delta_2 + 1)\delta_1 \end{pmatrix}}_{N(\delta)} \eta.$$

Sorting for common powers leads to

$$\xi = \delta_1 \left[\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \eta + \xi \right] + \delta_1^2 \left[\begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix} \eta + \xi \right] + \delta_1^2\delta_2 \left[\begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix} \eta - \xi \right] +$$

$$+ \delta_2 \left[\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \eta + 2\xi \right] + \delta_1\delta_2 \left[\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \eta - \xi \right] + \delta_2^2 \left[\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \eta - \xi \right].$$

In a first step we pull out the parameters as

$$\xi = w_1 + w_4, \ w_1 = \delta_1 z_1, \ w_4 = \delta_2 z_4$$

$$z_1 = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \eta + \xi + \delta_1 \left[\begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix} \eta + \xi \right] + \delta_1\delta_2 \left[\begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix} \eta - \xi \right]$$

$$z_4 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \eta + 2\xi + \delta_1 \left[\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \eta - \xi \right] + \delta_2 \left[\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \eta - \xi \right].$$

We iterate with the second step as

$$\xi = w_1 + w_4, \ w_1 = \delta_1 z_1, \ w_2 = \delta_1 z_2, \ w_3 = \delta_1 z_3, \ w_4 = \delta_2 z_4, \ w_5 = \delta_2 z_5$$

$$z_1 = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \eta + \xi + w_2, \ z_2 = \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix} \eta + \xi + \delta_2 \left[\begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix} \eta - \xi \right]$$

$$z_4 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \eta + 2\xi + w_3 + w_5, \ z_3 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \eta - \xi, \ z_5 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \eta - \xi.$$

It just requires one last step to complete the separation as

$$\xi = w_1 + w_4, \ w_1 = \delta_1 z_1, \ w_2 = \delta_2 z_2, \ w_3 = \delta_1 z_3, \ w_4 = \delta_1 z_4, \ w_5 = \delta_2 z_5, \ w_6 = \delta_2 z_6$$

$$z_1 = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \eta + \xi + w_2, \ z_2 = \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix} \eta + \xi + w_6, \ z_6 = \begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix} \eta - \xi$$

$$z_4 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \eta + 2\xi + w_3 + w_5, \ z_3 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \eta - \xi, \ z_5 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \eta - \xi.$$

In matrix format this can be expressed as

$$
\begin{pmatrix} \xi \\ \hline z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} =
\left(\begin{array}{cc||cc|cc|cc|cc|cc|cc}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline\hline
2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline
-1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline
-1 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ \hline
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-2 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{array}\right)
\begin{pmatrix} \eta \\ \hline w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{pmatrix}
$$

$$
\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \hline w_4 \\ w_5 \\ w_6 \end{pmatrix} =
\left(\begin{array}{ccc|ccc}
\delta_1 I_2 & 0 & 0 & 0 & 0 & 0 \\
0 & \delta_1 I_2 & 0 & 0 & 0 & 0 \\
0 & 0 & \delta_1 I_2 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & \delta_2 I_2 & 0 & 0 \\
0 & 0 & 0 & 0 & \delta_2 I_2 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta_2 I_2
\end{array}\right)
\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ \hline z_4 \\ z_5 \\ z_6 \end{pmatrix}.
$$

The parameter-block turns out to be of size 12. We stress that, even for the rather simple case at hand, a less careful construction can easily lead to much larger-sized LFR's. For example with SISO realization techniques it is not difficult to find LFR's of the elements of D, N of sizes 4 and 3 respectively. If using the operations augmentation, inversion and multiplication one obtains an LFR with parameter-block of size $(4+4)+(4+3+3+3) = 21$. We will briefly comment on various aspects around the practical construction of LFR's in the next section.

6.3 Robust stability and the full block S-procedure

Consider the uncertain time-invariant system (6.1.1) with time-invariant parameter vector $\delta \in \boldsymbol{\delta}$. In Chapter 5 we derived a number of LMI feasibility tests that guarantee that $F(\delta)$ is Hurwitz for all $\delta \in \boldsymbol{\delta}$. Most of these tests were either conservative or of limited scope since in Chapter 5 we assumed F to be affine or the uncertainty region $\boldsymbol{\delta}$ to be box-shaped (cf. Proposition 5.3 and Theorem 5.7).

Using linear fractional representations, a non-conservative robustness test for the uncertain dynamical system (6.1.1) can be given as follows.

Theorem 6.7 *Let the uncertain dynamical system $\dot{x} = F(\delta)x$ be represented by the linear fractional representation*

$$
\begin{pmatrix} \dot{x} \\ z \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}, \qquad w = \Delta(\delta)z
$$

where Δ is linear for all $\delta \in \boldsymbol{\delta}$. Define the transfer function $T(s) = C(Is-A)^{-1}B+D$. Then the LFR is well posed and $F(\delta)$ is Hurwitz for all $\delta \in \boldsymbol{\delta}$ if and only if

$$\det(I - T(i\omega)\Delta(\delta)) \neq 0, \quad \text{for all } \omega \in \mathbb{R} \cup \{\infty\}, \ \delta \in \boldsymbol{\delta}$$

Proof. Suppose the LFR is well-posed for $\delta \in \boldsymbol{\delta}$. Then $F(\delta)$ is Hurwitz for all $\delta \in \boldsymbol{\delta}$ if and only if the characteristic polynomial

$$\det(Is-A-B\Delta(\delta)(I-D\Delta(\delta))^{-1}C) \neq 0, \quad \text{for all} \operatorname{Re}(s) \geq 0, \ \delta \in \boldsymbol{\delta}$$

Using the well-posedness of the LFR, the Schur complement is well defined and gives the equivalent condition that

$$\det \begin{pmatrix} Is-A & -B\Delta(\delta) \\ -C & I-D\Delta(\delta) \end{pmatrix} \neq 0, \quad \text{for all } \operatorname{Re}(s) \geq 0, \ \delta \in \boldsymbol{\delta}$$

Taking once more a Schur complement gives the equivalent condition

$$\det \left(I - [D+C(Is-A)^{-1}B]\Delta(\delta) \right) \neq 0, \quad \text{for all} \operatorname{Re}(s) \geq 0, \ \delta \in \boldsymbol{\delta}$$

By a nontrivial homotopy argument, this is equivalent to saying that

$$\det \left(I - [D+C(i\omega I-A)^{-1}B]\Delta(\delta) \right) \neq 0, \quad \text{for all } \omega \in \mathbb{R}, \ \delta \in \boldsymbol{\delta}$$

Now observe that the latter condition is well-posedness for $\omega = \infty$. $\blacksquare$

Testing robust stability therefore boils down to verifying a robust non-singularity condition for the matrices $T(i\omega)$, with $\omega \in \mathbb{R} \cup \{\infty\}$, and the matrices $\Delta(\delta)$ with $\delta \in \boldsymbol{\delta}$.

Although useful, the frequency condition on the transfer function T is difficult to verify for any $\delta \in \boldsymbol{\delta}$. We therefore wish to derive numerically verifiable conditions for which $F(\delta)$ is Hurwitz for all $\delta \in \boldsymbol{\delta}$. The LFR's that we derived in the previous section prove very useful for this. Suppose that $F(\delta) = H \star \Delta(\delta)$ is a linear fractional representation of $F(\delta)$ and let us denote by

$$\boldsymbol{\Delta}_c = \{\Delta(\delta) \mid \delta \in \boldsymbol{\delta}\} = \Delta(\boldsymbol{\delta}) \tag{6.3.1}$$

the collection of (complex or real valued) matrices $\Delta(\delta)$ that can be created by ranging δ over the uncertainty set $\boldsymbol{\delta}$. Recall that the quadratic stability test that we considered in Chapter 5 (Definition 5.2) amounts to finding $K \succ 0$ such that

$$\left. \begin{array}{c} H \star \Delta \quad \text{is well-posed and} \\ \begin{pmatrix} I \\ H \star \Delta \end{pmatrix}^* \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \begin{pmatrix} I \\ H \star \Delta \end{pmatrix} \prec 0 \quad \text{for all } \Delta \in \boldsymbol{\Delta}_c. \end{array} \right\} \tag{6.3.2}$$

Here is a fundamental result which allows to design relaxations of (6.3.2) that are computationally tractable with LMI's.

Theorem 6.8 (Concrete full block S-procedure)
If there exist a multiplier $P = P^\top = \begin{pmatrix} Q & S \\ S^* & R \end{pmatrix}$ such that

$$\begin{pmatrix} \Delta \\ I \end{pmatrix}^* P \begin{pmatrix} \Delta \\ I \end{pmatrix} \succcurlyeq 0 \ \text{for all} \ \Delta \in \mathbf{\Delta}_c \tag{6.3.3}$$

and

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^\top \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^\top P \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \prec 0 \tag{6.3.4}$$

then (6.3.2) holds. The converse holds in case $\mathbf{\Delta}_c$, defined in (6.3.1), is compact.

As will be explained below it is elementary to show that (6.3.3) and (6.3.4) are sufficient for (6.3.2) whereas the proof of necessity is more difficult. Before we embed this concrete version of the S-procedure into a more versatile abstract formulation we intend to address the practical benefit of this reformulation. By introducing the *multiplier* or *scaling matrix P*, Theorem 6.8 expresses that one can equivalently rephrase (6.3.2) involving a multivariable rational function into the LMI conditions (6.3.3) and (6.3.4). Let us introduce, for the purpose of clarity, the obviously convex set of all multipliers that satisfy the infinite family of LMI's (6.3.3):

$$\mathscr{P}_{\text{all}} := \left\{ P = \begin{pmatrix} Q & S \\ S^* & R \end{pmatrix} \mid \begin{pmatrix} \Delta \\ I \end{pmatrix}^* P \begin{pmatrix} \Delta \\ I \end{pmatrix} \succcurlyeq 0 \ \text{for all} \ \Delta \in \mathbf{\Delta}_c \right\}.$$

Testing (6.3.2) then simply amounts to finding an element in $\mathscr{P}_{\text{all}}$ which satisfies the LMI (6.3.4). Unfortunately, the actual implementation of this test requires a description of $\mathscr{P}_{\text{all}}$ through finitely many LMI's, or at least a close approximation thereof. The need for such an approximation is the fundamental reason for conservatism in typical robust stability and robust performance tests in control!

Let us assume that $\mathscr{P}_i \subset \mathscr{P}_{\text{all}}$ is an inner approximation with an LMI description. Then the computation of $P \in \mathscr{P}_i$ with (6.3.4) is computationally tractable. If the existence of such a multiplier can be verified, it is clear that (6.3.2) has been verified. On the other hand let $\mathscr{P}_{\text{all}} \subset \mathscr{P}_o$ be an outer approximation. If one can computationally confirm that there does not exist any $P \in \mathscr{P}_o$ with (6.3.4), it is guaranteed that (6.3.2) is not true. We stress that the non-existence of $P \in \mathscr{P}_i$ or the existence of $P \in \mathscr{P}_o$ satisfying (6.3.2) does not allow to draw any conclusion without any additional knowledge about the quality of the inner or outer approximations respectively. In the sequel we discuss a selection of possible choices for inner approximations that have been suggested in the literature.

- **Spectral-norm multipliers.** The simplest and crudest inner approximation is obtained by neglecting any structural properties of the matrices in $\mathbf{\Delta}_c$ and just exploiting a known bound on their spectral norm. Indeed suppose that $r > 0$ is any (preferably smallest) number with

$$\|\Delta\| \leq r \ \text{for all} \ \Delta \in \mathbf{\Delta}_c.$$

Since

$$\|\Delta\| \leq r \quad \Longleftrightarrow \quad \frac{1}{r}\Delta^*\Delta \preccurlyeq rI \quad \Longleftrightarrow \quad \begin{pmatrix} \Delta \\ I \end{pmatrix}^* \begin{pmatrix} -1/r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \Delta \\ I \end{pmatrix} \succcurlyeq 0$$

we can choose the set

$$\mathscr{P}_{\mathrm{n}} := \left\{ \tau \begin{pmatrix} -1/r & 0 \\ 0 & r \end{pmatrix} \mid \tau \geq 0 \right\}$$

as an inner approximation, and checking (6.3.4) amounts to solving a one-parameter LMI problem.

- **Full-block multipliers.** For a more refined inner approximation let us assume that the set $\mathbf{\Delta}_c$ is described as

$$\mathbf{\Delta}_c = \mathrm{conv}(\mathbf{\Delta}_g) = \mathrm{conv}\{\Delta^1, \ldots, \Delta^N\} \ \text{ with } \ \mathbf{\Delta}_g = \{\Delta^1, \ldots, \Delta^N\}$$

as finitely many generators. Then define

$$\mathscr{P}_{\mathrm{full}} := \left\{ P = \begin{pmatrix} Q & S \\ S^* & R \end{pmatrix} \mid Q \preccurlyeq 0, \ \begin{pmatrix} \Delta \\ I \end{pmatrix}^* P \begin{pmatrix} \Delta \\ I \end{pmatrix} \succcurlyeq 0 \ \text{ for all } \ \Delta \in \mathbf{\Delta}_g \right\}.$$

Since any $P \in \mathscr{P}_{\mathrm{full}}$ has a north-west block which is negative semi-definite, the mapping

$$\Delta \mapsto \begin{pmatrix} \Delta \\ I \end{pmatrix}^* P \begin{pmatrix} \Delta \\ I \end{pmatrix} \ \text{ is concave,}$$

and hence positivity of its values at the generators $\Delta \in \mathbf{\Delta}_g$ implies positivity for all $\Delta \in \mathbf{\Delta}_c$. We conclude that $\mathscr{P}_{\mathrm{full}} \subset \mathscr{P}_{\mathrm{all}}$. On the other hand, $\mathscr{P}_{\mathrm{full}}$ is described by finitely many LMI's; hence searching for $P \in \mathscr{P}_{\mathrm{full}}$ satisfying (6.3.4) is a standard LMI problem that can be easily implemented.

By linearity of the mapping $\delta \mapsto \Delta(\delta)$ we infer for convex finitely generated parameter sets that

$$\mathbf{\Delta}_c = \mathrm{conv}\{\Delta(\delta^j) \mid j = 1, \ldots, N\} \ \text{ whenever } \ \boldsymbol{\delta} = \mathrm{conv}\{\delta^1, \ldots, \delta^N\}.$$

This means that the generators in $\mathbf{\Delta}_g$ are given by $\Delta^j = \Delta(\delta^j)$. In particular, for parameter boxes (products of intervals) defined as

$$\boldsymbol{\delta} = \{\delta \in \mathbb{R}^p : \ \delta_j \in [a_j, b_j], \ j = 1, \ldots, p\}$$

we can use the representation

$$\boldsymbol{\delta} = \mathrm{conv}(\boldsymbol{\delta}_{\mathrm{g}}) \ \text{ with } \ \boldsymbol{\delta}_{\mathrm{g}} := \{\delta \in \mathbb{R}^p : \ \delta_j \in \{a_j, b_j\}, \ j = 1, \ldots, p\}.$$

All these choices do not depend upon a specific structure of $\Delta(\delta)$ and hence allow for non-diagonal parameter-blocks.

- **Extended full-block multipliers.** Can we be more specific if $\boldsymbol{\delta}$ is a box generated by $\boldsymbol{\delta}_{\mathrm{g}}$ and the parameter-block $\Delta(\delta)$ is diagonal, as in (6.1.6)? With the column partition $(E_1 \ \cdots \ E_m) = I$ of the identity matrix conformable with that of $\Delta(\delta)$, we infer that

$$\Delta(\delta) = \sum_{k=1}^{p} E_k [\delta_k I] E_k^\top.$$

Let us then define

$$\mathscr{P}_{\mathrm{fe}} := \left\{ P \mid E_k^\top Q E_k \preccurlyeq 0, \ k=1,\ldots,p, \ \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}^* P \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} \succcurlyeq 0, \ \delta \in \pmb{\delta}_{\mathrm{g}} \right\}.$$

Due to $E_k^\top Q E_k \preccurlyeq 0$, $k=1,\ldots,p$, we conclude for any $P \in \mathscr{P}_{\mathrm{fe}}$ that the mapping

$$\delta \mapsto \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^* & R \end{pmatrix} \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} \quad \text{is concave in } \delta_k \text{ for } k=1,\ldots,p.$$

It is easily seen that this implies again that $\mathscr{P}_{\mathrm{fe}} \subset \mathscr{P}_{\mathrm{all}}$. We stress that $\mathscr{P}_{\mathrm{full}} \subset \mathscr{P}_{\mathrm{fe}}$ so that $\mathscr{P}_{\mathrm{fe}}$ is a potentially better inner approximation of $\mathscr{P}_{\mathrm{all}}$ which leads to less conservative numerical results. This comes at the expense of a more complex description of $\mathscr{P}_{\mathrm{fe}}$ since it involves a larger number of LMI's. This observation leads us to the description of the last inner approximation of $\mathscr{P}_{\mathrm{all}}$ in our non-exhaustive list.

- **Diagonal multipliers.** Suppose that the parameter space $\pmb{\delta}$ is a box in which $a_j \leq \delta_j \leq b_j$, $j=1,\ldots,p$. Let $m_j = (a_j+b_j)/2$ denote the mid-point of the interval $[a_j,b_j]$ and $d_j = (b_j-a_j)/2$ half of its diameter. We then infer that

$$\delta_j \in [a_j,b_j] \iff (\delta_j-m_j)^2 \leq d_j^2 \iff \begin{pmatrix} \delta_j \\ 1 \end{pmatrix}^* \underbrace{\begin{pmatrix} -1 & m_j \\ m_j & d_j^2-m_j^2 \end{pmatrix}}_{\begin{pmatrix} q_j & s_j \\ s_j^* & r_j \end{pmatrix}} \begin{pmatrix} \delta_j \\ 1 \end{pmatrix} \succcurlyeq 0.$$

Since

$$\begin{pmatrix} E_k & 0 \\ 0 & E_k \end{pmatrix}^T \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} = \begin{pmatrix} \sum_{l=1}^m E_k^T E_l[\delta_l I]E_l^T \\ E_k^T \end{pmatrix} = \begin{pmatrix} \delta_k I \\ I \end{pmatrix} E_k^T$$

(due to $E_k^T E_l = 0$ for $k \neq l$ and $E_k^T E_k = I$) we infer for $D_j \succcurlyeq 0$ that

$$\begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}^\top \sum_{j=1}^p \begin{pmatrix} E_j & 0 \\ 0 & E_j \end{pmatrix} \begin{pmatrix} q_j D_j & s_j D_j \\ s_j D_j & r_j D_j \end{pmatrix} \begin{pmatrix} E_j & 0 \\ 0 & E_j \end{pmatrix}^\top \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} =$$

$$= \sum_{j=1}^p E_j \begin{pmatrix} \delta_j I \\ I \end{pmatrix}^\top \begin{pmatrix} q_j D_j & s_j D_j \\ s_j^* D_j & r_j D_j \end{pmatrix} \begin{pmatrix} \delta_j I \\ I \end{pmatrix} E_j^\top =$$

$$= \sum_{j=1}^p E_j D_j E_j^\top \begin{pmatrix} \delta_j \\ 1 \end{pmatrix}^\top \begin{pmatrix} q_j & s_j \\ s_j^* & r_j \end{pmatrix} \begin{pmatrix} \delta_j \\ 1 \end{pmatrix} \succcurlyeq 0.$$

The fact that δ_j is real can be expressed as

$$\delta_j^* + \delta_j = 0 \ \text{ or } \ \begin{pmatrix} \delta_j \\ 1 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_j \\ 1 \end{pmatrix} = 0.$$

Exactly the same computation as above reveals for any $G_j = G_j^*$ that

$$\begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}^* \sum_{j=1}^p \begin{pmatrix} E_j & 0 \\ 0 & E_j \end{pmatrix} \begin{pmatrix} 0 & G_j \\ -G_j & 0 \end{pmatrix} \begin{pmatrix} E_j & 0 \\ 0 & E_j \end{pmatrix}^\top \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} = 0.$$

This implies for the set of block-diagonal multipliers

$$\mathscr{P}_{\mathrm{d}} := \left\{ \sum_{j=1}^{p} \begin{pmatrix} E_j & 0 \\ 0 & E_j \end{pmatrix} \begin{pmatrix} q_j D_j & s_j D_j + G_j \\ s_j^* D_j - G_j & r_j D_j \end{pmatrix} \begin{pmatrix} E_j & 0 \\ 0 & E_j \end{pmatrix}^{\top} : D_j \succcurlyeq 0, \ G_j = G_j^* \right\}$$

that $\mathscr{P}_{\mathrm{d}} \subset \mathscr{P}_{\mathrm{all}}$. Note that the set $\mathscr{P}_{\mathrm{d}}$ is explicitly parametrized by finitely many linear matrix inequalities.

- **Mixtures.** Let us assume that $\Delta(\delta)$ is diagonal as in (6.1.6). Our discussion makes it possible to construct sets of multipliers that are mixtures of those that have been listed already, as illustrated for the real-parameter example

$$\mathbf{\Delta}_c = \{ \mathrm{diag}(\delta_1 I_2, \delta_2, \delta_3, \delta_4 I_4) : \ \delta_1 \in [-1,1], \ \delta_2 \in [2,4], \ |\delta_3| \leq 1 - \delta_4, \ \delta_4 \geq -1 \} .$$

One can then work with the multiplier set of all

$$P = \left(\begin{array}{cccc|cccc} -D_1 & 0 & 0 & 0 & G_1 & 0 & 0 & 0 \\ 0 & -D_2 & 0 & 0 & 0 & 3D_2 + G_2 & 0 & 0 \\ 0 & 0 & Q_{11} & Q_{12} & 0 & 0 & S_{11} & S_{12} \\ 0 & 0 & Q_{12}^* & Q_{22} & 0 & 0 & S_{21} & S_{22} \\ \hline -G_1 & 0 & 0 & 0 & D_1 & 0 & 0 & 0 \\ 0 & -3D_2 - G_2 & 0 & 0 & 0 & -2D_2 & 0 & 0 \\ 0 & 0 & S_{11}^* & S_{21}^* & 0 & 0 & R_{11} & R_{12} \\ 0 & 0 & S_{12}^* & S_{22}^* & 0 & 0 & R_{12}^* & R_{22} \end{array} \right)$$

with

$$D_1, \geq 0, \ D_2 \geq 0, \ G_1 = G_1^*, \ G_2 = G_2^* \ \text{ and } \ Q_{11} \leq 0, \ Q_{22} \leq 0$$

as well as

$$\begin{pmatrix} \delta_3 & 0 \\ 0 & \delta_4 I_4 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}^* \left(\begin{array}{cc|cc} Q_{11} & Q_{12} & S_{11} & S_{12} \\ Q_{12}^* & Q_{22} & S_{21} & S_{22} \\ \hline S_{11}^* & S_{21}^* & R_{11} & R_{12} \\ S_{12}^* & S_{22}^* & R_{12}^* & R_{22} \end{array} \right) \begin{pmatrix} \delta_3 & 0 \\ 0 & \delta_4 I_4 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \geq 0, \ (\delta_3, \delta_4) = \left\{ \begin{array}{l} (-2, -1) \\ (2, -1). \\ (0, 1) \end{array} \right.$$

Of course it is straightforward to construct other block structures which all might lead to different numerical results in actual computations.

With the various classes of multipliers we arrive at the chain of inclusions

$$\mathscr{P}_{\mathrm{n}} \subset \mathscr{P}_{\mathrm{d}} \subset \mathscr{P}_{\mathrm{full}} \subset \mathscr{P}_{\mathrm{fe}} \subset \mathscr{P}_{\mathrm{all}}$$

which is briefly characterized as allowing the reduction of conservatism at the expense of increase in complexity of their descriptions. As the main distinction, we stress that the number of LMI's to describe $\mathscr{P}_{\mathrm{d}}$ grows linearly in the number p of parameters, whereas that for parametrizing $\mathscr{P}_{\mathrm{full}}$ grows exponentially in p. This shows the practical relevance of allowing for mixed block structures to be able to reduce conservatism while avoiding the explosion of computational complexity.

Remark 6.9 In practice, instead of just testing feasibility of (6.3.4) for some multiplier class $P \in \boldsymbol{P}$, it is rather suggested to choose some $\varepsilon > 0$ and to infimize γ over $P \in \boldsymbol{P}$ with

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^* & R \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^* \begin{pmatrix} Q_p & S_p \\ S_p^* & R_p \end{pmatrix} \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \leq \gamma \begin{pmatrix} \varepsilon I & 0 \\ 0 & I \end{pmatrix}.$$

In this fashion the largest eigenvalue of the left-hand side is really pushed to smallest possible values. If there exists some $P \in \boldsymbol{P}$ with (6.3.4) then, trivially, there exists some $\varepsilon > 0$ such that the infimal value γ_* is negative. It requires only slight modifications of the arguments to follow in order to show that

$$\Delta \star H \text{ is well-posed and } \begin{pmatrix} I \\ \Delta \star H \end{pmatrix}^* P_p \begin{pmatrix} I \\ \Delta \star H \end{pmatrix} \leq \gamma_* I \text{ for all } \Delta \in \boldsymbol{\Delta}_c.$$

Hence the value of γ_* provides an indication of distance to robust performance failure.

6.4 Some practical issues on LFR construction

6.4.1 Order reduction

It is of crucial importance to construct LFR's with a reasonably sized parameter-block. If $F(\delta) = H \star \Delta(\delta)$, then it is desirable to systematically construct an LFR $H_r \star \Delta_r(\delta)$ whose parameter-block $\Delta_r(\delta)$ is minimal in dimension and such that $H \star \Delta(\delta) = H_r \star \Delta_r(\delta)$ for all δ for which both LFR's are well-posed. Unfortunately, this minimal representation problem does not admit a computationally tractable solution.

One method to reduce dimension is the *channel-by-channel scheme*. Consider the LFR

$$\begin{pmatrix} \xi \\ \hline z_1 \\ \vdots \\ z_p \end{pmatrix} = \begin{pmatrix} A & B_1 & \cdots & B_p \\ \hline C_1 & D_{11} & \cdots & D_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ C_p & D_{p1} & \cdots & D_{pp} \end{pmatrix} \begin{pmatrix} \eta \\ \hline w_1 \\ \vdots \\ w_p \end{pmatrix}, \quad w_j = \delta_j z_j, \ j = 1, \ldots, p. \tag{6.4.1}$$

For all channels j, associate with this LFR (via the substitution $w_j \to x_j$, $z_j \to \dot{x}_j$) the linear time-invariant system

$$\begin{pmatrix} \xi \\ \hline z_1 \\ \vdots \\ \dot{x}_j \\ \vdots \\ z_p \end{pmatrix} = \begin{pmatrix} A & B_1 & \cdots & B_j & \cdots & B_p \\ \hline C_1 & D_{11} & \cdots & D_{1j} & \cdots & D_{1p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ C_j & D_{j1} & \cdots & D_{jj} & \cdots & D_{jp} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ C_p & D_{p1} & \cdots & D_{mj} & \cdots & D_{pp} \end{pmatrix} \begin{pmatrix} \eta \\ \hline w_1 \\ \vdots \\ x_j \\ \vdots \\ w_p \end{pmatrix}.$$

Whenever non-minimal, this realization can be reduced to a controllable and observable system to arrive at

$$
\left(\begin{array}{c}
\xi \\ \hline
z_1 \\
\vdots \\
\dot{\hat{x}}_j \\
\vdots \\
z_p
\end{array}\right)
=
\left(\begin{array}{c|ccccc}
A & B_1 & \cdots & \hat{B}_j & \cdots & B_p \\ \hline
C_1 & D_{11} & \cdots & \hat{D}_{1j} & \cdots & D_{1p} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\hat{C}_j & \hat{D}_{j1} & \cdots & \hat{D}_{jj} & \cdots & \hat{D}_{jp} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
C_p & D_{p1} & \cdots & \hat{D}_{pj} & \cdots & D_{pp}
\end{array}\right)
\left(\begin{array}{c}
\eta \\ \hline
w_1 \\
\vdots \\
\hat{x}_j \\
\vdots \\
w_p
\end{array}\right)
$$

with $\hat{x}_j$ of smaller dimension than x_j. If sequentially performing this reduction step for all $j = 1, \ldots, p$ we arrive at the LFR

$$
\left(\begin{array}{c}
\xi \\ \hline
\hat{z}_1 \\
\vdots \\
\hat{z}_j \\
\vdots \\
\hat{z}_p
\end{array}\right)
=
\left(\begin{array}{c|ccccc}
A & \hat{B}_1 & \cdots & \hat{B}_j & \cdots & \hat{B}_p \\ \hline
\hat{C}_1 & \hat{D}_{11} & \cdots & \hat{D}_{1j} & \cdots & \hat{D}_{1p} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\hat{C}_j & \hat{D}_{j1} & \cdots & \hat{D}_{jj} & \cdots & \hat{D}_{jp} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\hat{C}_p & \hat{D}_{p1} & \cdots & \hat{D}_{pj} & \cdots & \hat{D}_{pp}
\end{array}\right)
\left(\begin{array}{c}
\eta \\ \hline
\hat{w}_1 \\
\vdots \\
\hat{w}_j \\
\vdots \\
\hat{w}_p
\end{array}\right)
\tag{6.4.2}
$$

with $\hat{z}_j$ and $\hat{w}_j$ of smaller size than z_j and w_j. A little reflection convinces us that (6.4.1) is indeed equivalent to (6.4.2) for all δ for which both LFR's are well-posed. Note that the algebraic variety of δ for which the original LFR and the new LFR are well-posed are generally different. for an illustration of how these schemes work in practice we consider an example.

6.4.2   Approximation

If exact reduction of a given LFR $F(\delta) = H \star \Delta(\delta)$ is hard or impossible, one might consider constructing an approximation $H_a \star \Delta_a(\delta)$ with smaller-sized $\Delta_a(\delta)$. Most LFR's are typically used for parameters δ that reside in some a priori given set $\boldsymbol{\delta}$. If both the original and the approximate LFR's are well-posed for all $\delta \in \boldsymbol{\delta}$, we can use

$$
\sup_{\delta \in \boldsymbol{\delta}} \| H \star \Delta(\delta) - H_a \star \Delta_a(\delta) \|
$$

(with any matrix norm $\| \cdot \|$) as a measure for the distance of the two LFR's, and hence as an indicator for the approximation quality. We stress that the evaluation of the distance requires knowledge of the values $F(\delta) = H \star \Delta(\delta)$ at $\delta \in \boldsymbol{\delta}$. Given $F(\delta)$, this leads to the viable idea to construct LFR's by non-linear optimization. Among the various possibilities we would like to sketch one scheme that will be successfully applied to our example. Choose a parameter-block $\Delta_a(\delta)$ (of small size) which depends linearly on δ together with finitely many distinct points $\delta_1, \ldots, \delta_N \in \boldsymbol{\delta}$ (which should be well-distributed in $\boldsymbol{\delta}$). Then the goal is to find A_a, B_a, C_a, D_a such that

$$
\max_{j=1,\ldots,N} \left\| G(\delta_j) - \left(\begin{array}{cc} A_a & B_a \\ C_a & D_a \end{array}\right) \star \Delta_a(\delta_j) \right\|
\tag{6.4.3}
$$

is as small as possible. Clearly,

$$V_j = \begin{pmatrix} A_a & B_a \\ C_a & D_a \end{pmatrix} \star \Delta_a(\delta_j) - G(\delta_j) \iff$$

$$\iff \exists U_j: \quad V_j = A_a + U_j C_a - G(\delta_j), \quad U_j = B_a \Delta_a(\delta_j)(I - D_a \Delta_a(\delta_j))^{-1} \iff$$

$$\iff \exists U_j: \quad U_j[D_a \Delta_a(\delta_j) - I] + B_a \Delta_a(\delta_j) = 0, \quad A_a + U_j C_a - G(\delta_j) = V_j \iff$$

$$\iff \exists U_j: \quad U_j \begin{pmatrix} D_a \Delta_a(\delta_j) - I \\ C_a \end{pmatrix} + \begin{pmatrix} B\Delta_a(\delta_j) \\ A_a - G(\delta_j) \end{pmatrix} = \begin{pmatrix} 0 \\ V_j \end{pmatrix}.$$

Therefore we have to solve the nonlinear optimization problem

$$
\begin{aligned}
\text{minimize} \quad & \max_{j=1,\ldots,N} \|U_j C_a + A_a - G(\delta_j)\| \\
\text{subject to} \quad & U_j[D_a \Delta_a(\delta_j) - I] + B_a \Delta_a(\delta_j) = 0, \quad j = 1,\ldots,N.
\end{aligned}
\tag{6.4.4}
$$

in the variables $A_a, B_a, C_a, D_a, U_j, j = 1,\ldots,N$. It might be relevant to explicitly include the constraint that $D_a \Delta_a(\delta_j) - I$ has to be invertible. Moreover we stress that it is immediate to formulate variations which might considerably simplify the corresponding computations. For example if striving for LFR's which are exact in $\delta_1,\ldots,\delta_N$, one could solve the non-linear least-squares problem

$$
\inf_{\substack{A_a, B_a, C_a, D_a \\ U_j,\ j=1,\ldots,N}} \sum_{j=1}^{N} \left\| U_j \begin{pmatrix} D_a \Delta_a(\delta_j) - I \\ C_a \end{pmatrix} + \begin{pmatrix} B_a \Delta_a(\delta_j) \\ A_a - G(\delta_j) \end{pmatrix} \right\|_F.
\tag{6.4.5}
$$

Example 6.10 We continue Example 6.6 by constructing an LFR with nonlinear least-squares optimization. One easily verifies that the denominator polynomial $(\delta_2 - 1)(\delta_2 + \delta_1^2 + \delta_1 - 1)$ is nonzero on the box

$$\boldsymbol{\delta} = \{(\delta_1, \delta_2): \; -0.3 \le \delta_1 \le 0.3, \; -0.3 \le \delta_1 \le 0.3\}.$$

With $\Delta(\delta) = \mathrm{diag}(\delta_1, \delta_1, \delta_1, \delta_2, \delta_2, \delta_2)$ and the 49 points $\{-0.3 + 0.1j: \; j = 0,\ldots,6\}^2$ we solve (6.4.5) to achieve a cost smaller than $1.3 \cdot 10^{-9}$. The deviation (6.4.3) turns out to be not larger than $8 \cdot 10^{-5}$ on the denser grid $\{-0.3 + 0.01j: \; j = 0,\ldots,60\}^2$ with 3721 points. We have thus constructed an approximate LFR of considerably reduced order $(d_1, d_2) = (3, 3)$. It can actually be verified that

$$F(\delta) = \left(\begin{array}{cc|cccc} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 2 & 1 & 1 & 1 & 1 & 2 \\ -1 & 1 & 1 & 0 & 1 & 1 \\ -1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \star \begin{pmatrix} \delta_1 & 0 & 0 & 0 \\ 0 & \delta_1 & 0 & 0 \\ 0 & 0 & \delta_2 & 0 \\ 0 & 0 & 0 & \delta_2 \end{pmatrix}$$

which could not be reconstructed with any of the suggested techniques! The Examples 6.6-6.10 demonstrate that it might be difficult to reconstruct this small-sized LFR from the corresponding rational function with reasonable computational effort. A main lesson to be learnt here is that one should keep track of how large-sized LFR's are actually composed by smaller ones through interconnection in order not to loose essential structural information.

6.4.3 Approximate LFR's of non-rational functions.

The approximationn technique described in the previous subsection is only relying on the availability of $G(\delta_j)$ for $j = 1, \ldots, N$. This opens the path to determine LFR approximations for non-rational matrix valued mappings or even for mappings that are just defined through look-up tables. As we have seen, the direct construction of LFR's then requires the solution of a generally non-convex optimization problem.

Alternatively one could first interpolate the data matrices with multivariable polynomial or rational functions, and then construct an LFR of the interpolant. With multivariable polynomial matrices $D(\delta)$ and $N(\delta)$ of fixed degree and to-be-determined coefficients, this requires to solve the equation $D(\delta_j)^{-1}N(\delta_j) = G(\delta_j)$ which can be reduced to $D(\delta_j)G(\delta_j) - N(\delta_j) = 0$ (for all $j = 1, \ldots, N$) and hence turns out to be a linear system of equations in the coefficient matrices.

As a further alternative one can rely on polynomial or rational approximation. This amounts to finding polynomial matrices D, N which minimize $\max_{j=1,\ldots,N} \|D(\delta_j)^{-1}N(\delta_j) - G(\delta_j)\|$. For a fixed denominator polynomial matrix D and for a fixed degree of N this is clearly a convex optimization problem. Indeed, as an immediate consequence of the proof of Theorem 6.4, any such fixed denominator function can be parameterized as $\begin{pmatrix} A_a & B_a \\ C_a & D_a \end{pmatrix} \star \Delta(\delta) \star$ with *fixed* A_a, C_a and *free* B_a, D_a. Then it is obvious that (6.4.4) is a convex optimization problem which can be reduced to an LMI-problem if using any matrix norm whose sublevel sets admit an LMI-representation. For general rational approximation one has to rely on solving the non-convex problem (6.4.4) with *free* A_a, B_a, C_a, D_a, or one could turn to the multitude of existing alternatives such as Padé approximation for which we refer to the approximation literature.

6.4.4 Non-diagonal parameter-blocks

We have proved how to construct LFR's of rational matrices with diagonal parameter-blocks in a routine fashion. It is often possible to further reduce the size of LFR's with the help of full non-diagonal parameter-blocks which are only required to depend linearly on the parameters.

Example 6.11 At this point we disclose that the rational matrix in Examples 6.6-6.10 was actually constructed as

$$
F(\delta) = \left(\begin{array}{cc|ccc}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 \\
\hline
1 & 0 & 1 & 1 & 0 \\
-1 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 1
\end{array} \right) \star \left(\begin{array}{ccc}
\delta_1 & \delta_2 & \delta_1 \\
0 & \delta_1 & \delta_2 \\
0 & 0 & \delta_2
\end{array} \right)
$$

which is still smaller than all the LFR's constructed so far.

Example 6.12 Let us illustrate how to actually find such reduced-sized LFR's by means of the

example

$$F(\delta) = \frac{3\delta_1^2 - 2\delta_2}{1 - 4\delta_1 + 2\delta_1\delta_2 - \delta_2}.$$

A key role is played by finding suitable (matrix) factorizations. Indeed $\xi = F(\delta)\eta$ if and only if

$$\xi = (3\delta_1^2 - 2\delta_2)\eta + (4\delta_1 - 2\delta_1\delta_2 + \delta_2)\xi = \begin{pmatrix} \delta_1 & \delta_2 \end{pmatrix} \begin{pmatrix} 3\delta_1 & 4 - 2\delta_2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix} \iff$$

$$\iff \xi = w_1, \ w_1 = \begin{pmatrix} \delta_1 & \delta_2 \end{pmatrix} z_1, \ z_1 = \begin{pmatrix} 4\xi \\ -2\eta + \xi \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \delta_1 & \delta_2 \end{pmatrix} \begin{pmatrix} 3\eta \\ -2\xi \end{pmatrix} \iff$$

$$\iff \xi = w_1, \ w_1 = \begin{pmatrix} \delta_1 & \delta_2 \end{pmatrix} z_1, \ z_1 = \begin{pmatrix} 4\xi + w_2 \\ -2\eta + \xi \end{pmatrix}, \ w_2 = \begin{pmatrix} \delta_1 & \delta_2 \end{pmatrix} z_2, \ z_2 = \begin{pmatrix} 3\eta \\ -2\xi \end{pmatrix}$$

which leads to the LFR

$$\begin{pmatrix} \xi \\ \hline z_1 \\ \hline z_2 \end{pmatrix} = \left(\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 4 & 1 \\ -2 & 1 & 0 \\ \hline 3 & 0 & 0 \\ 0 & -2 & 0 \end{array} \right) \begin{pmatrix} \eta \\ \hline w_1 \\ w_2 \end{pmatrix}, \quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \left(\begin{array}{cc|cc} \delta_1 & \delta_2 & 0 & 0 \\ 0 & 0 & \delta_1 & \delta_2 \end{array} \right) \begin{pmatrix} z_1 \\ \hline z_2 \end{pmatrix}. \quad (6.4.6)$$

We observe that this LFR has the structure

$$\begin{pmatrix} \xi \\ \hline z \\ \hat{z} \end{pmatrix} = \left(\begin{array}{c|c} A & B \\ \hline LA & LB \\ C & D \end{array} \right) \begin{pmatrix} \eta \\ \hline w \end{pmatrix}, \quad w = \begin{pmatrix} \Delta(\delta) & \hat{\Delta}(\delta) \end{pmatrix} \begin{pmatrix} z \\ \hat{z} \end{pmatrix}$$

with $L = \begin{pmatrix} 0 & -2 & -4/3 \end{pmatrix}$ and is, since $\hat{z} = Lz$, equivalent to

$$\begin{pmatrix} \xi \\ z \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \eta \\ w \end{pmatrix}, \quad w = [\Delta(\delta) + \hat{\Delta}(\delta)L]z.$$

Therefore (6.4.6) can be compressed to

$$\begin{pmatrix} \xi \\ \hline z_1 \\ \hline z_2 \end{pmatrix} = \left(\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 4 & 1 \\ -2 & 1 & 0 \\ \hline 3 & 0 & 0 \end{array} \right) \begin{pmatrix} \eta \\ \hline w_1 \\ w_2 \end{pmatrix} \quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \left(\begin{array}{cc|c} \delta_1 & \delta_2 & 0 \\ 0 & -2\delta_2 & \delta_1 - 4/3\delta_2 \end{array} \right) \begin{pmatrix} z_1 \\ \hline z_2 \end{pmatrix}$$

with a parameter-block of size 2×3. Standard LFR's can be obtained with orders $(d_1, d_2) = (3, 1)$ or $(d_1, d_2) = (2, 2)$ which involves parameter-blocks of larger size 4×4.

6.5 Numerical solution of robust linear algebra problems

6.5.1 Relaxations for computing upper bounds

Let us assume that the decision variable is $v \in \mathbb{R}^n$, and suppose that

$$H(v) = \begin{pmatrix} A & B \\ C(v) & D(v) \end{pmatrix} \text{ with } C(v), D(v) \text{ depending affinely on } v,$$

and

$$P_p(v) = \begin{pmatrix} Q_p(v) & S_p \\ S_p^* & T_p U_p(v)^{-1} T_p^* \end{pmatrix} \text{ with } Q_p(v), U_p(v) \text{ depending affinely on } v.$$

Moreover let $\Delta(\delta)$ be a parameter-block which is linear in δ, and $\boldsymbol{\delta}$ is some subset of $\mathbb{R}^m$.

With the linear cost functional defined by $c \in \mathbb{R}^n$ we consider the following paradigm problem.

Problem Infimize $c^* v$ over all v which satisfy $U_p(v) \succ 0$ and

$$\det(I - A\Delta(\delta)) \neq 0, \quad \begin{pmatrix} I \\ \Delta(\delta) \star H(v) \end{pmatrix}^T P_p(v) \begin{pmatrix} I \\ \Delta(\delta) \star H(v) \end{pmatrix} < 0 \text{ for all } \delta \in \boldsymbol{\delta}. \quad (6.5.1)$$

Denote the optimal value by $\gamma_{\text{opt}} \in [-\infty, \infty]$.

In various circumstances we have made explicit how to construct a set $\boldsymbol{P}$ with an LMI description and such that

$$\begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}^* P \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} \succcurlyeq 0 \text{ for all } \delta \in \boldsymbol{\delta}. \quad (6.5.2)$$

This allows to consider the following relaxation.

Problem Infimize $c^* v$ over all v and $P \in \boldsymbol{P}$ with

$$U_p(v) \succ 0, \quad \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^* P \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ C(v) & D(v) \end{pmatrix}^* P_p(v) \begin{pmatrix} 0 & I \\ C(v) & D(v) \end{pmatrix} \prec 0.$$

Denote the optimal value by $\gamma_{\text{rel}} \in [-\infty, \infty]$.

Since $\boldsymbol{P}$ has an LMI description, the linearization lemma **??** reveals that we can compute γ_{rel} by solving a genuine LMI problem. The (simple) sufficiency conclusion in the full-block S-procedure implies that v is feasible for Problem 6.5.1 if v and $P \in \boldsymbol{P}$ are feasible for Problem 6.5.1. We can hence infer

$$\gamma_{\text{opt}} \leq \gamma_{\text{rel}},$$

including the cases $\gamma_{\text{opt}} \in \{-\infty, \infty\}$ and $\gamma_{\text{rel}} \in \{-\infty, \infty\}$ with the usual interpretations.

Unfortunately it is impossible to make a priori statements on the relaxation gap $\gamma_{\text{rel}} - \gamma_{\text{opt}}$ in the generality as discussed here. However, let us stress again that this gap can be possibly reduced by

employing a larger (superset) of $\boldsymbol{P}$, and that $\gamma_{\text{opt}} \leq \gamma_{\text{rel}}$ if $\boldsymbol{\delta}$ is compact and if replacing $\boldsymbol{P}$ with the set of all multipliers P that satisfy 6.5.2, just due to the full-block S-procedure. Summarizing, the choice of an increasing family of multipliers $\boldsymbol{P}_1 \subset \boldsymbol{P}_2 \subset \cdots$ leads to a family of LMI-relaxations with non-increasing optimal values.

We conclude this section by stressing that the developed techniques allow an immediate extension to multiple-objectives expressed by finitely many constraints as

$$\left(\begin{array}{c} I \\ \Delta(\delta) \star H(v) \end{array} \right)^T P_{p,j}(v) \left(\begin{array}{c} I \\ \Delta(\delta) \star H(v) \end{array} \right) \prec 0 \ \text{ for all } \ \delta \in \boldsymbol{\delta}, \ j = 1, \ldots, p.$$

These constraints are relaxed as follows: For all $j = 1, \ldots, p$, there exists $P_j \in \boldsymbol{P}$ with

$$\left(\begin{array}{cc} I & 0 \\ A & B \end{array} \right)^* P_j \left(\begin{array}{cc} I & 0 \\ A & B \end{array} \right) + \left(\begin{array}{cc} 0 & I \\ C(v) & D(v) \end{array} \right)^* P_{p,j}(v) \left(\begin{array}{cc} 0 & I \\ C(v) & D(v) \end{array} \right) \prec 0.$$

It is hence essential to exploit the extra freedom to relax any individual constraint with its individual multiplier in order to keep conservatism subdued. We remark as well that one could even allow the LFR and the class of multipliers used for each relaxation vary from constraint to constraint without introducing any extra complications.

6.5.2 When are relaxations exact?

We have seen in the previous section that multiplier relaxations typically cause a gap $\gamma_{\text{rel}} - \gamma_{\text{opt}}$. In this section we formulate a general principle about a numerical verifiable sufficient condition for the absences of this gap and hence for exactness of the relaxation.

One situation is simple: If the relaxation is strictly feasible and has optimal value $\gamma_{\text{rel}} = -\infty$. This certainly implies $\gamma_{\text{opt}} = -\infty$ and there is no relaxation gap. Let us hence assume from now on that $\gamma_{\text{rel}} > -\infty$.

The key is to consider the dual of the suggested relaxations. For this purpose we first apply the linearization lemma **??** in order to reformulate the relaxation as infimizing c^* over v and $P \in \boldsymbol{P}$ with

$$\left(\begin{array}{ccc} I & 0 & 0 \\ A & B & 0 \end{array} \right)^* P \left(\begin{array}{ccc} I & 0 & 0 \\ A & B & 0 \end{array} \right) + \left(\begin{array}{ccc} 0 & C(v)^*S_p^* & C(v)^*T_p \\ S_pC(v) & S_pD(v) + D(v)^*S_p^* + Q_p(v) & D(v)^*T_p \\ T_p^*C(v) & T_p^*D(v) & -U_p(v) \end{array} \right) \prec 0.$$

With the standard basis vectors $e_1, \ldots, e_n$ of $\mathbb{R}^n$ and with $e_0 = 0$ define

$$c_j = c(e_j), \ \ W_j = \left(\begin{array}{ccc} 0 & C(e_j)^*S_p^* & C(e_j)^*T_p \\ S_pC(e_j) & S_pD(e_j) + D(e_j)^*S_p^* + Q_p(e_j) & D(e_j)^*T_p \\ T_p^*C(e_j) & T_p^*D(e_j) & -U_p(e_j) \end{array} \right), \ \ j = 0, 1, \ldots, n.$$

Then we have to infimize c^*x over x and $P \in \boldsymbol{P}$ subject to

$$\left(\begin{array}{ccc} I & 0 & 0 \\ A & B & 0 \end{array} \right)^* P \left(\begin{array}{ccc} I & 0 & 0 \\ A & B & 0 \end{array} \right) + W_0 + \sum_j x_j W_j \leq 0.$$

For $\boldsymbol{\delta} = \text{conv}\{\delta^1, \ldots, \delta^N\}$ let us now consider the concrete class $\boldsymbol{P}$ of full block multipliers P which are just implicitly described by the (strictly feasible) constraints

$$\begin{pmatrix} I \\ 0 \end{pmatrix}^* P \begin{pmatrix} I \\ 0 \end{pmatrix} \le 0, \quad \begin{pmatrix} \Delta(\delta^j) \\ I \end{pmatrix}^* P \begin{pmatrix} \Delta(\delta^j) \\ I \end{pmatrix} \succcurlyeq 0 \quad \text{for all} \ \ j = 1, \ldots, N.$$

With Lagrange multipliers M, $\hat{M}$, M_j, $j = 1, \ldots, N$, the Lagrangian hence reads as

$$\sum_{j=1}^{n} c_j x_j + \left\langle \begin{pmatrix} I & 0 & 0 \\ A & B & 0 \end{pmatrix}^* P \begin{pmatrix} I & 0 & 0 \\ A & B & 0 \end{pmatrix}, M \right\rangle + \langle W_0, M \rangle + \sum_{j=1}^{n} x_j \langle W_j, M \rangle +$$

$$+ \left\langle \begin{pmatrix} I \\ 0 \end{pmatrix}^* P \begin{pmatrix} I \\ 0 \end{pmatrix}, \hat{M} \right\rangle - \sum_{j=1}^{N} \left\langle \begin{pmatrix} \Delta(\delta^j) \\ I \end{pmatrix}^* P \begin{pmatrix} \Delta(\delta^j) \\ I \end{pmatrix}, M_j \right\rangle$$

or, after 'sorting for primal variables',

$$\langle W_0, M \rangle + \sum_{j=1}^{n} (c_j + \langle W_j, M \rangle) x_j +$$

$$+ \left\langle P, \begin{pmatrix} I & 0 & 0 \\ A & B & 0 \end{pmatrix} M \begin{pmatrix} I & 0 & 0 \\ A & B & 0 \end{pmatrix}^* + \begin{pmatrix} I \\ 0 \end{pmatrix} \hat{M} \begin{pmatrix} I \\ 0 \end{pmatrix}^* + \sum_{j=1}^{N} \begin{pmatrix} \Delta(\delta^j) \\ I \end{pmatrix} M_j \begin{pmatrix} \Delta(\delta^j) \\ I \end{pmatrix}^* \right\rangle.$$

By standard duality we can draw the following conclusions:

(a) The primal is not strictly feasible (which just means $\gamma_{rel} = \infty$) iff there exist $M \succcurlyeq 0$, $\hat{M} \succcurlyeq 0$, $M_j \succcurlyeq 0$ with $\langle W_0, M \rangle \geqslant 0$, $\langle W_j, M \rangle = 0$, $j = 1, \ldots, N$ and

$$\begin{pmatrix} I & 0 & 0 \\ A & B & 0 \end{pmatrix} M \begin{pmatrix} I & 0 & 0 \\ A & B & 0 \end{pmatrix}^* + \begin{pmatrix} I \\ 0 \end{pmatrix} \hat{M} \begin{pmatrix} I \\ 0 \end{pmatrix}^* + \sum_{j=1}^{N} \begin{pmatrix} \Delta(\delta^j) \\ I \end{pmatrix} M_j \begin{pmatrix} \Delta(\delta^j) \\ I \end{pmatrix}^* = 0.$$

$$(6.5.3)$$

(b) If the primal is strictly feasible and has finite optimal value there exists $M \succcurlyeq 0$, $\hat{M} \succcurlyeq 0$, $M_j \succcurlyeq 0$ which maximize $\langle W_0, M \rangle$ under the constraints $\langle W_j, M \rangle + c_j = 0$, $j = 1, \ldots, N$ and (6.5.3). The optimal value of this Lagrange dual problem equals γ_{rel}.

Theorem 6.13 *If $\gamma_{rel} = \infty$ and if M in 1. has rank one then $\gamma_{opt} = \infty$. If $\gamma_{rel} < \infty$ and if M in 2. has rank one then $\gamma_{opt} = \gamma_{rel}$.*

It is convenient to summarize this result as follows: If there exists either an infeasibility certificate or a dual optimizer such that M has rank one then the relaxation is exact.

Proof. If M has rank one, it can be decomposed as $M = mm^*$. Let us partition m as $\text{col}(w, \xi, \xi_e)$ according to the columns of $(A \ B \ 0)$ and define $z = (A \ B \ 0)m = Aw + B\xi$. The essential point is to conclude from (6.5.3) that

$$w = \Delta(\delta^0)z \quad \text{for some} \ \ \delta^0 \in \boldsymbol{\delta}.$$

Indeed just by using the definitions we obtain with (6.5.3) the relation

$$
\begin{pmatrix} w \\ z \end{pmatrix} \begin{pmatrix} w^* & z^* \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} \hat{M} \begin{pmatrix} I & 0 \end{pmatrix} - \sum_{j=1}^{N} \begin{pmatrix} \Delta(\delta^j) \\ I \end{pmatrix} M_j \begin{pmatrix} \Delta(\delta^j) \\ I \end{pmatrix}^* = 0.
$$

From $zz^* = \sum_{j=1}^{N} M_j$ we infer that $z^*x = 0$ implies $M_j x = 0$ for all j such that there exist α_j with $M_j = z\alpha_j z^*$. If $z \neq 0$ have $\alpha_j \geq 0$ and $\sum_{j=1}^{N} \alpha_j = 1$. Now $wz^* = \sum_{j=1}^{N} \Delta(\delta^j) M_j = 0$ allows to conclude by right-multiplication with z and division by $z^*z \neq 0$ that

$$
w = \sum_{j=1}^{N} \Delta(\delta^j) z \alpha_j = \Delta \left(\sum_{j=1}^{N} \alpha_j \delta^j \right) z = \Delta(\delta^0) z \quad \text{with} \quad \delta^0 := \sum_{j=1}^{N} \alpha_j \delta^j \in \boldsymbol{\delta}.
$$

If $z = 0$ we infer $M_j = 0$ and hence $ww^* + \hat{M} = 0$ which implies $w = 0$ (and $\hat{M} = 0$) such that $w = \Delta(\delta^0) z$ holds for an arbitrary $\delta^0 \in \boldsymbol{\delta}$.

Let us now assume $\gamma_{\text{rel}} = \infty$. In can happen that $I - A\Delta(\delta^0)$ is singular. This implies that Problem 6.5.1 is not feasible and hence $\gamma_{\text{opt}} = \infty$.

Let us continue under the hypothesis that $I - A\Delta(\delta^0)$ is non-singular which implies $w = \Delta(\delta^0)(I - A\Delta(\delta^0))^{-1} B\xi$ (due to $z = Aw + B\xi$ and $w = \Delta(\delta^0)z$). We infer for all $x \in \mathbb{R}^n$:

$$
\begin{aligned}
\xi^* \begin{pmatrix} * \\ * \end{pmatrix}^* & \begin{pmatrix} Q(x) & S \\ S^* & TU(x)^{-1}T^* \end{pmatrix} \begin{pmatrix} I \\ D(x) + C(x)\Delta(\delta^0)(I - A\Delta(\delta^0))^{-1}B \end{pmatrix} \xi = \\
&= \begin{pmatrix} w \\ \xi \end{pmatrix}^* \begin{pmatrix} 0 & I \\ C(x) & D(x) \end{pmatrix}^* \begin{pmatrix} Q(x) & S \\ S^* & TU(x)^{-1}T^* \end{pmatrix} \begin{pmatrix} 0 & I \\ C(x) & D(x) \end{pmatrix} \begin{pmatrix} w \\ \xi \end{pmatrix} = \\
&= \max_{\eta} \begin{pmatrix} w \\ \xi \\ \eta \end{pmatrix}^* \begin{pmatrix} 0 & C(x)^*S_p^* & C(x)^*T_p \\ S_p C(x) & S_p D(x) + D(x)^*S_p^* + Q_p(x) & D(x)^*T_p \\ T_p^* C(x) & T_p^* D(x) & -U_p(x) \end{pmatrix} \begin{pmatrix} w \\ \xi \\ \eta \end{pmatrix} \succcurlyeq \\
&\succcurlyeq \begin{pmatrix} w \\ \xi \\ \xi_e \end{pmatrix}^* \begin{pmatrix} 0 & C(x)^*S_p^* & C(x)^*T_p \\ S_p C(x) & S_p D(x) + D(x)^*S_p^* + Q_p(x) & D(x)^*T_p \\ T_p^* C(x) & T_p^* D(x) & -U_p(x) \end{pmatrix} \begin{pmatrix} w \\ \xi \\ \xi_e \end{pmatrix} = \\
&\hspace{6cm} = m^* W_0 m + \sum_{j=1}^{n} x_j m^* W_j m.
\end{aligned}
$$

By hypothesis we have

$$
0 \leq \langle W_0, M \rangle + \sum_{j=1}^{n} x_j \langle W_j, M \rangle = m^* W_0 m + \sum_{j=1}^{n} x_j m^* W_j m \quad \text{for all} \quad x \in \mathbb{R}^n.
$$

From the above-derived chain of inequalities we can hence infer that no $x \in \mathbb{R}^n$ can ever be feasible for (6.5.1), which implies again that $\gamma_{\text{opt}} = \infty$ and thus $\gamma_{\text{opt}} = \gamma_{\text{rel}}$.

Let us now assume $\gamma_{\text{rel}} < \infty$. Since problem 6.5.1 is strictly feasible it is guaranteed that $I - A\Delta(\delta^0)$ is nonsingular, and the above-derived chain of inequalities is again true. Let us apply this chain of

inequalities for any x that is feasible for Problem 6.5.1 to infer

$$0 \succcurlyeq m^*W_0m + \sum_{j=1}^n x_j m^*W_j m.$$

Just due to $\langle W_0, M \rangle = \gamma_{\text{rel}}$ and $c_j = \langle W_j, M \rangle$ we can conclude

$$\sum_{j=1}^n c_j x_j = \gamma_{\text{rel}} - \langle W_0, M \rangle - \sum_{j=1}^n x_j \langle W_j, M \rangle = \gamma_{\text{rel}} - m^*W_0m - \sum_{j=1}^n x_j m^*W_j m$$

and hence $c^*x \geq \gamma_{\text{rel}}$. Since v was an abitrary feasible point of Problem 6.5.1 we infer $\gamma_* \geq \gamma_{\text{rel}}$ and thus $\gamma_* = \gamma_{\text{rel}}$. ∎

It is interesting to note that the proof reveals how one can construct a worst-case parameter uncertainty from a rank one dual multiplier! In a similar fashion it is possible to apply the novel principle to a whole variety of other problems which is reveals its charming character.

In one specific case with one-dimensional uncertainty one can in fact *always* construct a dual multiplier of rank one.

Theorem 6.14 *Suppose that $\boldsymbol{\delta} = \text{conv}\{\delta^1, \delta^2\}$. Then $\gamma_{rel} = \gamma_{opt}$.*

Proof. Let us first investigate the constraint (6.5.3) in the specific case $N = 2$. Left-multiplication with $\begin{pmatrix} I & -\Delta(\delta^1) \end{pmatrix}$ and right-multiplication with $\begin{pmatrix} I & -\Delta(\delta^2) \end{pmatrix}^*$ leads with

$$U := \begin{pmatrix} I & -\Delta(\delta^1) \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ A & B & 0 \end{pmatrix} \text{ and } V := \begin{pmatrix} I & -\Delta(\delta^2) \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ A & B & 0 \end{pmatrix}$$

to the relation $UMV^* + \hat{M} = 0$. This implies $UMV^* + V^*MU = -\hat{M} \leq 0$ and $UMV^* - V^*MU = 0$ which can be expressed as

$$\begin{pmatrix} U^* \\ V^* \end{pmatrix}^* \begin{pmatrix} 0 & -M \\ -M & 0 \end{pmatrix} \begin{pmatrix} U^* \\ V^* \end{pmatrix} \succcurlyeq 0, \quad \begin{pmatrix} U^* \\ V^* \end{pmatrix}^* \begin{pmatrix} 0 & M \\ -M & 0 \end{pmatrix} \begin{pmatrix} U^* \\ V^* \end{pmatrix} = 0. \quad (6.5.4)$$

Now choose an arbitrary $x \in \mathbb{R}^n$. Since $M \neq 0$ (???), we can apply Lemma **??** to infer the existence of a vector $m \neq 0$ (possibly depending on x) with

$$m^* \left(W_0 + \sum_{j=1}^n x_j W_j \right) m \geq \langle W_0 + \sum_{j=1}^n x_j W_j, M \rangle. \quad (6.5.5)$$

and

$$\begin{pmatrix} m^*U^* \\ m^*V^* \end{pmatrix}^* \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} m^*U^* \\ m^*V^* \end{pmatrix} \succcurlyeq 0, \quad \begin{pmatrix} m^*U^* \\ m^*V^* \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} m^*U^* \\ m^*V^* \end{pmatrix} = 0.$$

If we partition m again as $\mathrm{col}(w, \xi, \xi_e)$ and if we define $z = Aw + B\xi$ we infer $Um = w - \Delta(\delta^1)z$ and $Vm = w - \Delta(\delta^2)z$. If $Vm = 0$ we conclude $w = \Delta(\delta^2)z$. If $Vm \neq 0$ there exists, by Lemma **??**, some $\alpha \leq 0$ with $w - \Delta(\delta^1)z = \alpha w - \alpha \Delta(\delta^2)z$ or

$$w = \left(\frac{1}{1-\alpha}\Delta(\delta^1) - \frac{\alpha}{1-\alpha}\Delta(\delta^2) \right) z = \Delta \left(\frac{1}{1-\alpha}\delta^1 - \frac{\alpha}{1-\alpha}\delta^2 \right) z.$$

Therefore in both cases $w = \Delta(\delta^0)z$ for some $\delta^0 \in \mathrm{conv}\{\delta^1, \delta^2\}$.

If $\gamma_{\mathrm{rel}} = \infty$, we can infer $\langle W_0 + \sum_{j=1}^n x_j W_j, M \rangle \succcurlyeq 0$ and hence with (6.5.5) we obtain $m^* W_0 m + \sum_{j=1}^n x_j m^* W_j m \succcurlyeq 0$. This allows to finish the proof as above.

If $\gamma_{\mathrm{rel}} < \infty$, and if x is chosen feasible for Problem **??**, we infer again $0 \succcurlyeq m^* W_0 m + \sum_{j=1}^n x_j m^* W_j m$. Now we exploit (6.5.5) to conclude $0 \succcurlyeq \langle W_0 + \sum_{j=1}^n x_j W_j, M \rangle$ or $c^* x \succcurlyeq \gamma_{\mathrm{rel}}$. We conclude again that $\gamma_{\mathrm{opt}} \geq \gamma_{\mathrm{rel}}$, and thus equality. ∎

Remarks. This is a novel result on the absence of a relaxation gap for robust semi-definite programs with implicitly described full-block multipliers. Only slight modifications of our arguments lead to the same insights for block-diagonal multipliers, which reveals that we have provided generalizations of the result from [**?**] and [**?**]. It is important to stress that full-block multipliers are in general expected to lead to less conservative results. We can conclude, however, that the one parameter case allows the restriction to the diagonal multiplier class without introducing conservatism. In case of complex uncertainties as appearing in SSV-theory we will be able to easily derive similar insights in a considerable more general setting in the sections to follow.

6.6 Further reading

We refer to [64] for a variety of operations and configurations on linear fractional representations. The practical construction of LFR's is supported by a very helpful, professionally composed and freely available Matlab LFR-Toolbox developed by Francois Magni. See [14, 27].

6.7 Exercises

Exercise 1
Suppose that $F(\delta) = D(\delta)^{-1}N(\delta)$ with multivariable polynomial matrices $D(\delta)$ and $N(\delta)$ such that $D(0)$ is invertible. If $(D \ -N)$ admits an LFR with parameter-block $\Delta(\delta)$, prove that G admits as well an LFR with the same parameter-block. Formulate a corresponding result for right-factorizations $F(\delta) = N(\delta)D(\delta)^{-1}$.

Exercise 2

Determine an LFR of

$$F(\delta) = \frac{-2\delta_2 + 3\delta_1^2 - 2\delta_1\delta_2 + 3\delta_1^3 - 2\delta_2^2 + 3\delta_1^2\delta_2}{1 - 3\delta_1 - 3\delta_1\delta_2 - 4\delta_1^2 + 2\delta_1^2\delta_2 + 2\delta_1\delta_2^2 - \delta_2^2}.$$

Is it possible to reduce the size of the resulting LFR through channel-by-channel reduction? Show that $1 + \delta_1 + \delta_2$ is a common factor of numerator and denominator. Use this insight to construct a smaller sized LFR.

Exercise 3
Suppose $0 \in \mathbf{\Delta}_c$. Derive exact LMI test for verifying the following two conditions:

- $\begin{pmatrix} I - \Delta M \\ N \end{pmatrix}$ has full column rank for all $\Delta \in \mathbf{\Delta}_c$.

- $N - M\Delta$ has full column rank for all $\Delta \in \mathbf{\Delta}_c$ (under hypothesis $0 \in \mathbf{\Delta}_c$).

Exercise 4
Suppose we are given a system $\dot{x} = Ax + Bw$, $z = Cx + Dw$ with

$$A = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 4 & -3 \\ 1 & -3 & -1 & -3 \\ 0 & 4 & 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(a) Let all coefficients of C deviate from their nominal values by 100%.

   (i) Determine the worst-case H_∞-norm. (Please argue how this can be done!)
   (ii) Compute the worst case H_∞-norm with a common quadratic storage function.
   (iii) Can you explain the difference?

(b) Now suppose that only the elements of A deviate from their nominal value by 5%. Let us try to compute a bound on the H_∞ norm with a common quadratic storage function.

   (i) If describing this uncertainty as $A \in \text{conv}\{A_1, \ldots, A_N\}$, how many generators N do you need? Is the direct approach as in the previous exercise expected to be tractable?

   (ii) Determine an LFR: Find a representation of the uncertain system as

$$\begin{pmatrix} \dot{x} \\ z_1 \\ z \end{pmatrix} = \begin{pmatrix} A_0 & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} w \\ w_1 \\ w \end{pmatrix}, \quad z_1 = \text{diag}(\delta_1 I, \ldots, \delta_{12} I) w_1.$$

   (iii) Consider the LMI problem of minimizing γ such that

$$X \succ 0, \quad \left(\begin{array}{ccc} I & 0 & 0 \\ A_0 & B_1 & B_2 \\ \hline 0 & I & 0 \\ C_1 & D_{11} & D_{12} \\ \hline 0 & 0 & I \\ C_2 & D_{21} & D_{22} \end{array} \right) \left(\begin{array}{cc|cc|cc} 0 & X & 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -Q & S & 0 & 0 \\ 0 & 0 & S^T & Q & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array} \right) \left(\begin{array}{ccc} I & 0 & 0 \\ A_0 & B_1 & B_2 \\ \hline 0 & I & 0 \\ C_1 & D_{11} & D_{12} \\ \hline 0 & 0 & I \\ C_2 & D_{21} & D_{22} \end{array} \right) \prec 0$$

where $Q = \mathrm{diag}(Q_1, \ldots, Q_{12}) \succeq 0$ and $S = \mathrm{diag}(S_1, \ldots, S_{12})$ is skew-symmetric. Why does this compute a bound on the worst-case H_∞-norm? (Don't forget stability.)

(iv) Compute such an optimal bound for the given system.

Exercise 5 (MIMO circle criterion and extensions)
Consider the nonlinear system

$$\dot{x} = Ax + Bw, \quad z = Cx, \quad w = \Delta(t, z) \tag{6.7.1}$$

where $\Delta : \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^k$ is any Lipschitz continuous function that satisfies, for two matrices K, L, the multi-variable sector condition

$$[\Delta(t, z) - Kz]^\top [\Delta(t, z) - Lz] \leq 0 \quad \text{for all } (t, z) \in \mathbb{R} \times \mathbb{R}^l. \tag{6.7.2}$$

(a) Find a symmetric matrix P such that

$$\left(\begin{array}{c} \Delta(z, t) \\ z \end{array} \right)^T P \left(\begin{array}{c} \Delta(z, t) \\ z \end{array} \right) \geq 0 \quad \text{for all } z, t.$$

(b) Formulate an LMI test for proving global exponential stability of (6.7.1).

(c) With $M(s) = C(sI - A)^{-1}B$ define $G(s) = (I - LM(s))(I - KM(s))^{-1}$. Show that the LMI you found has a solution if and only if

$$A + BKC \text{ is stable and } G(i\omega)^* + G(i\omega) \succ 0 \text{ for all } \omega \in \mathbb{R}.$$

Is stability of A required for your arguments?

Hint: try a congruence transformation with $\begin{pmatrix} I & 0 \\ KC & I \end{pmatrix}$.

(d) Consider now the saturation nonlinearity

$$\Delta(t, z) = \begin{cases} 1 & \text{for } z \geq 1 \\ z & \text{for } |z| \leq 1 \\ -1 & \text{for } z \leq -1 \end{cases}.$$

Show that it satisfies the sector condition (7.8.2) with $K = 0$ and $L = 1$.

(e) Is (6.7.1) with the saturation nonlinearity and

$$C(sI - A)^{-1}B = G(s) = \frac{4}{(s+1)(\frac{1}{2}s + 1)(\frac{1}{3}s + 1)}$$

globally exponentially stable?

(f) Let us now suppose that the system is excited by an L_2-disturbance d as

$$\dot{x} = Ax + B(w+d), \quad z = Cx, \quad w = \Delta(t,z).$$

Use the multiplier τP with $\tau \geq 0$ to set up an LMI optimization problem which minimizes a guaranteed bound γ on the L_2-gain of the nonlinear system with input d and output e.

(g) Compute such an optimal bound for the saturation nonlinearity in 4. and the system in 5.

Exercise 6 (Exactness of standard S-procedure)
Consider the functions $f_j : \mathbb{R}^n \to \mathbb{R}$ with $j = 0, 1, \ldots, m$. The S-procedure tries to give verifiable characterizations for the validity of the implication

$$x \in \mathbb{R}^n, \;\; f_1(x) \leq 0, \ldots, f_m(x) \leq 0 \;\; \Rightarrow \;\; f_0(x) \geq 0.$$

Note that geometrically this condition is equivalent to $\{x \in \mathbb{R}^n : f_0(x) \geq 0\}$ covering $\{x \in \mathbb{R}^n : f_j(x) \leq 0, \; j = 1, \ldots, m\}$. In terms of optimization, the implication holds if and only if

$$\inf_{f_j(x) \leq 0, \, j=1,\ldots,m} f_0(x) \;\; \text{is non-negative.} \tag{6.7.3}$$

The purpose of the following exercises is to relate this property to the following relaxation:

$$\exists \, \tau_1, \ldots, \tau_m \geq 0 : \;\; f_0(x) + \tau_1 f_1(x) + \cdots + \tau_m f_m(x) \geq 0 \quad \text{for all } \; x \in \mathbb{R}^n. \tag{6.7.4}$$

As a general hint one should recall the general discussion of Lagrange duality.

(a) Prove that (6.7.4) implies (6.7.3) without any hypotheses.

(b) Suppose that all f_j are convex and that there exists some $x_0 \in \mathbb{R}^n$ with $f_j(x_0) < 0$ for $j = 1, \ldots, m$. Show that (6.7.3) is equivalent to (6.7.4).

(c) Let f_j be quadratic functions represented as

$$f_j(x) = \begin{pmatrix} x \\ 1 \end{pmatrix}^* F_j \begin{pmatrix} x \\ 1 \end{pmatrix} \quad \text{with} \quad F_j = \begin{pmatrix} Q_j & s_j \\ s_j^* & r_j \end{pmatrix} \quad \text{for } \; j = 0, 1, \ldots, m.$$

Show that (6.7.4) can be verified by solving an LMI.
Hint: Prove and use the following auxiliary fact:

$$\begin{pmatrix} x \\ 1 \end{pmatrix}^* F \begin{pmatrix} x \\ 1 \end{pmatrix} \geq 0 \;\; \text{for all } \; x \in \mathbb{R}^n \quad \text{if and only if} \quad F \geq 0.$$

(d) Consider two quadratic functions f_j, $j = 0, 1$ ($m = 1$) and suppose there exists some x_0 with $f_1(x_0) < 0$. Prove that (6.7.3) implies (6.7.4) (without any convexity hypothesis).
*Hints: Show that the value of (6.7.3) equals $\inf\{z^* F_0 z : z \in \mathbb{R}^{n+1}, \, z^* F_1 z \leq 0\}$. Then prove that the latter equals $\inf\{\langle Z, F_0 \rangle : Z \geq 0, \, \langle Z, F_1 \rangle \leq 0\}$ by using Lemma **??** for two real matrices. Finally apply Lagrange dualization to this LMI problem.*

(e) Let $f_j : \mathbb{C}^n \to \mathbb{R}$ for $j = 0, 1, 2$ be again quadratic functions defined with complex data matrices F_0, F_1, F_2. Suppose that there exists some $x_0 \in \mathbb{C}^n$ with $f_1(x_0) < 0$ and $f_2(x_0) < 0$. Prove that (6.7.3) implies (6.7.4) (without any convexity hypothesis).

Hint: Follow the proof of the previous exercise and apply Lemma **??** *for three complex matrices.*

Chapter 7

Integral quadratic constraints

One of the main considerations in control is to study how signals are processed by dynamical systems and how this processing can be influenced in order to achieve a certain desired behavior. For that purpose one has to specify the signals (time series, trajectories) of interest. This is done by deciding on the set of values taken by the signals (such as $\mathbb{R}^n$) and on the time set on which they are defined (such as the full time axis $\mathbb{R}$, the half axis $[0, \infty)$ or the corresponding discrete time versions $\mathbb{Z}$ and $\mathbb{N}$). In our exposition, a dynamical system is then nothing but a mapping that assigns to a certain input signal some output signal. Very often, such mappings are defined by differential equations with fixed initial conditions or by an integral equation, which amounts to considering systems with a specific and explicit description or representation.

The purpose of this chapter is to discuss robust stability and performance of system interconnections in a rather abstract setting. It is our major goal to develop the theory of integral quadratic constraints which provides a unifying framework that covers the more classical approach based on stability multipliers. Particular emphasis is put on stability and performance characterizations that can be rephrased in terms of an LMI optimization problem.

7.1   Basic notions

Let us first describe the concrete ingredients that are required to develop the general theory. We assume some familiarity with elementary concepts of functional analysis, such as normed and inner product spaces and linear operators defined thereon. An excellent source for a precise exposition of most mathematical concepts in this chapter is [**?**] while [**?**] covers all the required somewhat advanced topics in a complete manner. From an engineering perspective it is helpful to consult [**?**] in which, in particular, some of the measure theoretic concepts are discussed in an elementary yet insightful fashion.

7.1.1 Signal spaces

The basis for defining signal spaces is formed by $\mathscr{L}^n$, the linear space of all time-functions or signals $x : [0,\infty) \to \mathbb{R}^n$ which are Lebesgue-measurable. Without bothering too much about the exact definition, one should recall that all continuous or piece-wise continuous signals are contained in $\mathscr{L}^n$. Moreover, without causing harm for our discussion, we ignore the fact that signals in $\mathscr{L}^n$ are only defined almost everywhere.

For any $x \in \mathscr{L}^n$ one can calculate the energy integral

$$\|x\| := \sqrt{\int_0^\infty \|x(t)\|^2 \mathrm{d}t}$$

which is either finite or infinite. The collection of all signals with finite energy is typically denoted by

$$\mathscr{L}_2^n := \{x \in \mathscr{L}^n \mid \|x\| < \infty\}.$$

It can be shown that $\mathscr{L}_2^n$ is a linear space, that $\|\cdot\|$ is a norm on $\mathscr{L}_2^n$, and that $\mathscr{L}_2^n$ is complete or a Banach space. The quantity $\|x\|$ is often called $\mathscr{L}_2$-norm (or energy) of the signal x.

Actually $\mathscr{L}_2^n$ admits an additional structure. Indeed, let us define the bilinear form

$$\langle x, y \rangle = \int_0^\infty x(t)^\top y(t) \mathrm{d}t$$

on $\mathscr{L}_2^n \times \mathscr{L}_2^n$. Bilinearity means that $\langle \cdot, y \rangle$ is linear for each $y \in \mathscr{L}_2^n$ and $\langle x, \cdot \rangle$ is linear for each $x \in \mathscr{L}_2^n$. It is not difficult to see that $\langle \cdot, \cdot \rangle$ defines an inner product on $\mathscr{L}_2^n$ and that the energy and the inner product are related in the standard fashion by $\|x\|^2 = \langle x, x \rangle$. As a complete inner product space $\mathscr{L}_2^n$ is in fact a Hilbert space.

For any $x \in \mathscr{L}_2^n$ one can determine the Fourier transform $\hat{x}$ which is a $\mathbb{C}^n$-valued function defined on the imaginary axis $\mathbb{C}^0$ and which has the property that

$$\int_{-\infty}^\infty \hat{x}(i\omega)^* \hat{x}(i\omega) \mathrm{d}\omega \quad \text{is finite.}$$

A fundamental results in the theory of the Fourier transformation on $\mathscr{L}_2$-spaces, Parseval's theorem, states that

$$\int_0^\infty x(t)^\top y(t) \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^\infty \hat{x}(i\omega)^* \hat{y}(i\omega) \mathrm{d}\omega.$$

For more advanced concepts and a detailed discussion on the relation to the so-called Hardy spaces the reader is referred to [**?**, **?**, 64].

Roughly speaking, stability of systems is related to the property that it maps any input signal of bounded energy into an output signal which also has bounded energy. Since it is desired to deal with unstable systems, we cannot confine ourselves to signals with finite energy only. This is the motivation for introducing a larger class of signals that have finite energy on finite intervals only.

For that purpose it is convenient to define, for each $T > 0$, the *truncation operator* which assigns to any signal $x \in \mathscr{L}^n$ the signal x_T that is identical to x on $[0, T]$ and that vanishes on (T, ∞):

$$x_T(t) := \begin{cases} x(t) & \text{for } t \in [0, T] \\ 0 & \text{for } t \in (T, \infty). \end{cases}$$

Note that $\mathscr{L}^n$ is a linear space and that truncation is a linear operator on that space with the additional property that $(x_T)_T = x_T$ for all $T > 0$. Hence truncation is a projection.

Now it is straightforward to define the so-called extended space $\mathscr{L}_{2e}^n$, consisting of all signals $x \in \mathscr{L}^n$ such that x_T has finite energy for all $T > 0$:

$$\mathscr{L}_{2e}^n := \{ x \in \mathscr{L}^n \mid x_T \in \mathscr{L}_2^n \text{ for all } T > 0 \}.$$

Hence any $x \in \mathscr{L}_{2e}^n$ has the property that

$$\|x_T\| = \int_0^T \|x(t)\|^2 \mathrm{d}t \text{ is finite for every } T > 0.$$

We observe that $\|x_T\|$ does not decrease if T increases. Therefore, $\|x_T\|$ viewed as a function of T either stays bounded for $T \to \infty$ and then converges, or it is unbounded and then it diverges to ∞. For any $x \in \mathscr{L}_{2e}^n$ we can hence conclude that $\|x_T\|$ is bounded for $T \to \infty$ if and only if x is contained in $\mathscr{L}_2^n$. Moreover,

$$x \in \mathscr{L}_2^n \text{ implies } \|x\| = \lim_{T \to \infty} \|x_T\|.$$

For the interpretation of some stability results to follow, we like to stress that the spaces $\mathscr{L}_2^n$ and $\mathscr{L}_{2e}^n$ are indistinguishable on finite intervals. More precisely,

$$(\mathscr{L}_2^n)_T = \{ x_T \mid x \in \mathscr{L}_2^n \} = \{ x_T \mid x \in \mathscr{L}_{2e}^n \} = (\mathscr{L}_{2e}^n)_T = \mathscr{L}_2[0, T] \quad \text{for all } T > 0,$$

with the latter equation being interpreted as a Hilbert-space isomorphism.

Example 7.1 The signal defined by $x(t) = e^t$ is contained in $\mathscr{L}_{2e}$ but not in $\mathscr{L}_2$. The signal defined by $x(0) = 0$ and $x(t) = 1/t$ for $t > 0$ is not contained in $\mathscr{L}_{2e}$. In general, since continuous or piecewise continuous signals are bounded on $[0, T]$ for every $T > 0$, they are all contained in $\mathscr{L}_{2e}^n$.

7.1.2 Systems

A *dynamical system S* is a mapping

$$S : \mathscr{L}_{2e}^k \longrightarrow \mathscr{L}_{2e}^l$$

which takes any input $u \in \mathscr{L}_{2e}^k$ into the output $y = S(u) \in \mathscr{L}_{2e}^l$. The system S is called *linear* if the map is, and the output is then often denoted by Su (leaving out the brackets in the notation). A general system S is said to be *causal* if it satisfies

$$S(u)_T = S(u_T)_T \text{ for all } T > 0, u \in \mathscr{L}_{2e}^k.$$

It is easily seen that this mathematical definition matches the intuitive notion of causality: If u and v are two input signals that are identical on $[0, T]$, $u_T = v_T$, then $S(u)$ and $S(v)$ are also identical on the same time-interval, $S(u)_T = S(v)_T$. In other words, future values of an input do not have any effect on the past values of the corresponding output.

Example 7.2 Consider the linear time-invariant (LTI) system

$$\begin{cases} \dot{x} = Ax + Bu, \quad x(0) = x_0 \\ y = Cx + Du \end{cases} \tag{7.1.1}$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times k}$, $C \in \mathbb{R}^{l \times n}$, $D \in \mathbb{R}^{l \times k}$ and $x_0 \in \mathbb{R}^n$. A standard fact in the theory of differential equations reveals that any $u \in \mathscr{L}_{2e}^k$ leads to a unique response $y \in \mathscr{L}_{2e}^l$. Therefore (7.1.1) defines a system $S : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^l$. The variation-of-constants formula

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t) \ \text{ for } \ t \geq 0$$

reveals that S is causal. If $x_0 = 0$ then S is linear.

Our main interest in this abstract setting is to characterize stability of systems and their interconnections. Among the many possibilities to define stability, those based on the gain or the incremental gain have turned out to be of prominent importance.

Definition 7.3 The $\mathscr{L}_2$-*gain* $\|S\|$ of the system S is the infimal real number $\gamma \geq 0$ for which there exist some real γ_0 with

$$\|S(u)_T\| \leq \gamma \|u_T\| + \gamma_0 \ \text{ for all } \ T > 0, \ u \in \mathscr{L}_{2e}^k. \tag{7.1.2}$$

If $\|S\|$ is finite then S is said to be bounded or to have a finite $\mathscr{L}_2$-gain.

It is elementary to show that $\|S\| \geq 0$, $\|\alpha S\| = |\alpha| \|S\|$ ($\alpha \in \mathbb{R}$), $\|S + T\| \leq \|S\| + \|T\|$ and $\|ST\| \leq \|S\|\|T\|$ for bounded systems S and T. However $\|\cdot\|$ only defines a semi-norm since $\|S\|$ can vanish even if $S \neq 0$.

For inputs with finite energy, the property (7.1.2) implies, in particular, that

$$\|S(u)\| \leq \gamma \|u\| + \gamma_0 \ \text{ for all } \ u \in \mathscr{L}_2^k. \tag{7.1.3}$$

Therefore S maps inputs of finite energy into outputs of finite energy, with an explicit bound of the output energy that is affine in the input energy. If S is causal, the converse is true as well and (7.1.3) implies (7.1.2). Indeed, for $u \in \mathscr{L}_{2e}^k$ we infer by causality that $\|S(u)_T\| = \|S(u_T)_T\| \leq \|S(u_T)\|$; since $u_T \in \mathscr{L}_2^k$ we can use (7.1.3) to conclude $\|S(u_T)\| \leq \gamma \|u_T\| + \gamma_0$ which proves (7.1.2).

For causal systems S, the $\mathscr{L}_2$-gain $\|S\|$ is hence equal to the infimal $\gamma \geq 0$ for which there exists γ_0 such that (7.1.3) holds true. In other words, stability properties of causal systems can be characterized in terms of their action on finite energy signals only.

If S is linear one can prove that (7.1.2) actually implies that

$$\|S(u)_T\| \leq \gamma \|u_T\| \ \text{ for all } \ T > 0, \ u \in \mathscr{L}_{2e}^k. \tag{7.1.4}$$

Hence, for linear systems $\|S\|$ and the concept of boundedness can be defined by setting $\gamma_0 = 0$ without loss of generality. Consequently, $\|S\|$ actually equals the minimal γ for which (7.1.4) is true, which can as well be expressed as

$$\|S\| = \sup_{T>0, u \in \mathscr{L}_{2e}^k, \|u_T\| \neq 0} \frac{\|S(u)_T\|}{\|u_T\|}.$$

In the sequel we will make tacit use of

$$\|S(u)_T\| \leq \|S\| \|u_T\| \text{ for all } T > 0, \ u \in \mathscr{L}_{2e}^k \ \text{ or } \ \|S(u)\| \leq \|S\| \|u\| \text{ for all } u \in \mathscr{L}_2^k.$$

Example 7.4 In continuing Example 7.2, one can prove that the output of (7.1.1) has finite energy for all $u \in \mathscr{L}_2^k$ if and only if the corresponding transfer matrix $C(sI - A)^{-1}B + D$ has all its poles in the open left-half plane. If (A,B) is stabilizable and (A,C) is detectable, this is equivalent to A being Hurwitz. If (7.1.1) maps $\mathscr{L}_2^k$ into $\mathscr{L}_2^l$, it has finite $\mathscr{L}_2$-gain which is know to be equal to the H_∞-norm of the transfer matrix $C(sI - A)^{-1}B + D$.

Example 7.5 With the function $\phi : \mathbb{R} \to \mathbb{R}$ let us define the *static nonlinear system*

$$S(u)(t) := \phi(u(t)) \ \text{ for } \ t \geq 0. \tag{7.1.5}$$

The function ϕ is said to satisfy a **sector condition** defined by real numbers $\alpha \leq \beta$ if its graph is located in the conic sector enclosed by the lines $\{(x,y) \in \mathbb{R}^2 \mid y = \beta x\}$ and $\{(x,y) \in \mathbb{R}^2 \mid y = \alpha x\}$. One can express this property as

$$\phi(0) = 0 \ \text{ and } \ \alpha \leq \frac{\phi(x)}{x} \leq \beta \ \text{ for } \ x \in \mathbb{R} \setminus \{0\}.$$

Alternatively, $\phi(x) - \alpha x$ and $\beta x - \phi(x)$ have the same sign for all $x \in \mathbb{R}$, which translates into $(\phi(x) - \alpha x)(\beta x - \phi(x)) \geq 0$ or

$$\begin{pmatrix} \phi(x) \\ x \end{pmatrix}^\top \begin{pmatrix} -1 & \frac{1}{2}(\alpha + \beta) \\ \frac{1}{2}(\alpha + \beta) & -\alpha\beta \end{pmatrix} \begin{pmatrix} \phi(x) \\ x \end{pmatrix} \geq 0 \ \text{ for all } \ x \in \mathbb{R}. \tag{7.1.6}$$

With $\gamma = \max\{|\alpha|, |\beta|\}$ it is easily checked that

$$|\phi(x)| \leq \gamma |x| \ \text{ for all } \ x \in \mathbb{R},$$

which in turn implies that $S : \mathscr{L}_{2e} \to \mathscr{L}_{2e}$ is causal and that $\|S\| \leq \gamma$.

Proof. If $x > 0$ we infer $\phi(x) \leq \beta x$ and hence $|\phi(x)| \leq |\beta| |x|$ which implies $|\phi(x)| \leq \gamma |x|$ for $\gamma = \max\{|\alpha|, |\beta|\}$. This implies $\phi(0) = 0$. A similar argument shows $|\phi(x)| \leq |\alpha| |x| \leq \gamma |x|$ for $x < 0$. Hence

$$|\phi(x)| \leq \gamma |x| \text{ for all } x \in \mathbb{R}.$$

If $u \in \mathscr{L}_{2e}$ then

$$S(u_T)(t) = \phi(u_T(t)) = \left\{ \begin{array}{ccc} \phi(u(t)) & \text{for} & t \in [0,T] \\ 0 & \text{for} & t > T \end{array} \right\} = S(u_T)_T(t)$$

which proves causality. Moreover we have

$$\|S(u_T)\|^2 = \int_0^T |\phi(u(t))|^2 \, dt \leq \gamma^2 \int_0^T |u(t)|^2 \, dt \leq = \gamma^2 \|u_T\|^2.$$

Since the choice of $u \in \mathscr{L}_{2e}$ was arbitrary we have $\|S\| \leq \gamma$.   ∎

Sometimes it is more convenient to characterize the $\mathscr{L}_2$-gain of a system by squared norms, as is possible without loss of generality.

Lemma 7.6 *The $\mathscr{L}_2$-gain of S equals the infimal $\gamma \geq 0$ for which there exists some γ_0 with*

$$\|S(u)_T\|^2 \leq \gamma^2 \|u_T\|^2 + \gamma_0 \text{ for all } T > 0, \ u \in \mathscr{L}_{2e}^k. \tag{7.1.7}$$

Proof. Note that (7.1.7) implies

$$\|S(u)_T\| \leq \sqrt{\gamma^2 \|u_T\|^2 + \gamma_0} \leq \sqrt{\gamma^2 \|u_T\|^2 + |\gamma_0|} \leq \gamma \|u_T\| + \sqrt{|\gamma_0|}$$

for all $T > 0$, $u \in \mathscr{L}_{2e}^k$. Therefore the infimum defined in the lemma is not smaller than $\|S\|$. Now suppose $\|S\| < \gamma$. Then one can choose $\varepsilon \in (0,1)$ with $(1+\varepsilon)\|S\| < \gamma$ and there exists γ_0 such that, for all $T > 0$ and $u \in \mathscr{L}_{2e}^k$, $(1+\varepsilon)\|S(u)_T\| \leq \gamma \|u_T\| + \gamma_0$ or

$$(1+\varepsilon)^2 \|S(u)_T\|^2 \leq \left(\begin{array}{c} 1 \\ \|u_T\| \end{array} \right)^\top \left(\begin{array}{cc} \gamma_0^2 & \gamma_0 \gamma \\ \gamma_0 \gamma & \gamma^2 \end{array} \right) \left(\begin{array}{c} 1 \\ \|u_T\| \end{array} \right).$$

Since $\gamma^2 < (1+\varepsilon)^2 \gamma^2$ and as a consequence of the Schur lemma, there exists some $\widehat{\gamma_0}$ with

$$\left(\begin{array}{cc} \gamma_0^2 & \gamma_0 \gamma \\ \gamma_0 \gamma & \gamma^2 \end{array} \right) \preccurlyeq \left(\begin{array}{cc} \widehat{\gamma_0} & 0 \\ 0 & (1+\varepsilon)^2 \gamma^2 \end{array} \right).$$

Therefore

$$(1+\varepsilon)^2 \|S(u)_T\|^2 \leq \left(\begin{array}{c} 1 \\ \|u_T\| \end{array} \right)^\top \left(\begin{array}{cc} \widehat{\gamma_0} & 0 \\ 0 & (1+\varepsilon)^2 \gamma^2 \end{array} \right) \left(\begin{array}{c} 1 \\ \|u_T\| \end{array} \right) = (1+\varepsilon)^2 \gamma^2 \|u_T\|^2 + \widehat{\gamma_0}$$

and hence $\|S(u)_T\|^2 \leq \gamma^2 \|u_T\|^2 + \widehat{\gamma_0}/(1+\varepsilon)^2$ for all $T > 0$ and $u \in \mathscr{L}_{2e}^k$. This implies that the infimum defined in the lemma is not larger than γ and, in turn, not larger than $\|S\|$.   ∎

For nonlinear systems it is often more useful to compare the distance of two different input signals u, v with the distance of the corresponding output signals $S(u)$, $S(v)$. This leads to the notion of *incremental $\mathscr{L}_2$-gain.*

Definition 7.7 The *incremental $\mathscr{L}_2$-gain* $\|S\|_i$ of the system $S : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^l$ is defined as the infimal γ such that
$$\|S(u)_T - S(v)_T\| \leq \gamma \|u_T - v_T\| \quad \text{for all} \ \ T > 0, \ u,v \in \mathscr{L}_{2e}^k.$$
If $\|S\|_i < \infty$ then S is said to have finite incremental $\mathscr{L}_2$-gain or *incrementally bounded*.

Again it is not hard to show that $\|S\|_i \geq 0$, $\|\alpha S\|_i = |\alpha| \|S\|_i$ ($\alpha \in \mathbb{R}$), $\|S + T\|_i \leq \|S\|_i + \|T\|_i$ and $\|ST\|_i \leq \|S\|_i \|T\|_i$. However $\|\cdot\|_i$ is certainly not a norm since, for example, constant nonzero systems have incremental gain zero. Note that, in general, we have
$$\|S\|_i = \sup_{T>0, u,v \in \mathscr{L}_{2e}^k, \|u_T - v_T\| \neq 0} \frac{\|S(u)_T - S(v)_T\|}{\|u_T - v_T\|}.$$

If S is causal, we can again infer that $\|S\|_i$ equals the minimal $\gamma \geq 0$ such that
$$\|S(u) - S(v)\| \leq \gamma \|u - v\| \quad \text{for all} \ \ u,v \in \mathscr{L}_2^k, \tag{7.1.8}$$
and thus
$$\|S\|_i = \sup_{u,v \in \mathscr{L}_2^k, \|u-v\| \neq 0} \frac{\|S(u) - S(v)\|}{\|u - v\|}.$$

Example 7.8 Let the function ϕ in Example 7.5 satisfy the incremental sector condition
$$\begin{pmatrix} \phi(x_1) - \phi(x_2) \\ x_1 - x_2 \end{pmatrix}^\top \begin{pmatrix} -1 & \frac{1}{2}(\alpha + \beta) \\ \frac{1}{2}(\alpha + \beta) & -\alpha\beta \end{pmatrix} \begin{pmatrix} \phi(x_1) - \phi(x_2) \\ x_1 - x_2 \end{pmatrix} \geq 0 \ \ \text{for all} \ \ x_1, x_2 \in \mathbb{R}. \tag{7.1.9}$$
We have seen that this can be expressed by
$$\alpha \leq \frac{\phi(x_1) - \phi(x_2)}{x_1 - x_2} \leq \beta \ \ \text{for all} \ \ x_1, x_2 \in \mathbb{R}, \ x_1 \neq x_2, \tag{7.1.10}$$
which boils down to imposing the bounds α and β on the slope of ϕ if it is differentiable. With $\gamma = \max\{|\alpha|, |\beta|\}$ one can infer that
$$|\phi(x_1) - \phi(x_2)| \leq \gamma |x_1 - x_2| \ \ \text{for all} \ \ x_1, x_2 \in \mathbb{R}.$$

Hence the mapping defined by (7.1.5) has finite incremental $\mathscr{L}_2$-gain which is bounded by γ. Note, however, that one cannot guarantee S to have finite $\mathscr{L}_2$-gain; just consider $\phi(x) = 1$ which maps the $\mathscr{L}_2$-function $u(t) = 0$ into $\phi(u(t)) = 1$ not contained in $\mathscr{L}_2$. As revealed by this example, the key difficulty is the fact that $\phi(0) \neq 0$ which implies $S(0) \notin \mathscr{L}_2$.

It just follows from the definitions that
$$\|S\| \leq \|S\|_i \ \ \text{in case that} \ \ S(u_0) \in \mathscr{L}_2^l \ \ \text{for some} \ \ u_0 \in \mathscr{L}_2^k.$$

Hence, systems with a finite incremental $\mathscr{L}_2$-gain which map at least one finite energy signal back into $\mathscr{L}_2^l$ do also have a finite $\mathscr{L}_2$-gain. Systems with $S(0) = 0$ form prominent examples. In general, the converse is not true. However,
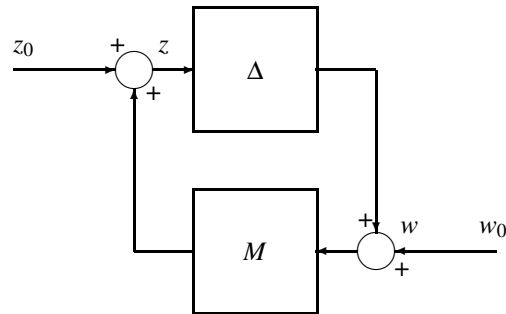$$|S\| = \|S\|_i \ \ \text{if} \ S \ \text{is linear}.$$

Figure 7.1: Specific feedback interconnection

7.2 Robust input-output stability

In general, robustness analysis is nothing but an investigation of the sensitivity of a relevant system property against (possibly large) perturbations that are known to belong to an a priori specified class of perturbations. In general, one can argue that robustness questions form center stage in most natural sciences, in engineering and in mathematics. In control, many robustness properties can conveniently be rephrased as stability or performance properties of interconnected systems in which specific components are perturbed. The purpose of this section is to develop tools for guaranteeing the stability of an interconnection of a linear time-invariant system against rather general classes of system perturbations or uncertainties.

7.2.1 A specific feedback interconnection

In previous chapters we have investigated the stability of interconnections if the system components are linear time-invariant. We established how to reduce stability analysis to an investigation of the particular feedback interconnection as depicted in Figure 7.1. In this chapter the main concern is to extend the earlier developed stability analysis techniques to such interconnections in which Δ is allowed to be a time-invariant or time-varying system which might be linear or non-linear. We develop conditions which are based on relatively simple arguments involving quadratic constraints but which result in far-reaching computational techniques for the specialization to so-called integral quadratic constraints.

Let us consider the interconnection in Figure 7.1 for systems

$$M : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^l \ \text{ and } \ \Delta : \mathscr{L}_{2e}^l \to \mathscr{L}_{2e}^k \ \text{ which are causal.}$$

This configuration translates into the algebraic relations $z = M(w) + d$, $w = \Delta(z)$ or

$$z - M\Delta(z) = d, \ \ w = \Delta(z). \tag{7.2.1}$$

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Here the signal $d \in \mathscr{L}_{2e}^l$ is considered to be an external disturbance of the interconnection, while z (and w) are viewed as its response to this external stimulus.

The feedback interconnection is said to be well-posed if for each disturbance $d \in \mathscr{L}_{2e}^l$ there exists a unique response $z \in \mathscr{L}_{2e}^k$ with

$$(I - M\Delta)(z) = z - M\Delta(z) = d$$

such that z depends causally on d. This is captured in the following formal definition.

Definition 7.9 The interconnection (7.2.1) as depicted in Figure 7.1 is *well-posed* if the system $I - M\Delta : \mathscr{L}_{2e}^l \to \mathscr{L}_{2e}^l$ has a causal inverse.

The verification of well-posedness often boils down to the application of results from the theory of differential equations, as illustrated in the following example.

Example 7.10 Recall from Example 7.2 that $M : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^k$ defined by

$$\begin{cases} \dot{x} = Ax + Bu \qquad x(0) = x_0 \\ y = Cx \end{cases}$$

is causal. Moreover suppose $\Delta : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^k$ is given by

$$\Delta_v(z)(t) := \phi_v(z_v(t)) \ \text{ for } \ t \geq 0, \ \ v = 1, \ldots, k,$$

where $\phi_v : \mathbb{R} \to \mathbb{R}$ are continuously differential functions which satisfy the sector condition

$$\begin{pmatrix} \phi_v(x) \\ x \end{pmatrix}^\top \begin{pmatrix} -1 & \frac{\alpha_v + \beta_v}{2} \\ \frac{\alpha_v + \beta_v}{2} & -\alpha_v \beta_v \end{pmatrix} \begin{pmatrix} \phi_v(x) \\ x \end{pmatrix} \geq 0 \ \text{ for all } \ x \in \mathbb{R}, \ \ v = 1, \ldots, k \qquad (7.2.2)$$

with constants $\alpha_v \leq \beta_v$. Observe that Δ can be addressed as a static diagonal nonlinearity. In view of Example 7.5, the system Δ is causal and its $\mathscr{L}_2$-gain is bounded by $\max\{|\alpha_1|, |\beta_1|, \ldots, |\alpha_k|, |\beta_k|\}$.

The map which takes z into $d = (I - M\Delta)(z)$ is described by $\dot{x} = Ax + B\Delta(z)$, $d = z - Cx$, $x(0) = x_0$ which is equivalent to

$$\begin{cases} \dot{x} = Ax + B\Delta(Cx + d), \qquad x(0) = x_0, \\ z = Cx + d. \end{cases} \qquad (7.2.3)$$

Checking well-posedness requires to show that, for every $d \in \mathscr{L}_{2e}^k$, the initial value problem in (7.2.3) has a unique solution on $[0, \infty)$ such that $z \in \mathscr{L}_{2e}^k$. It is a consequence of classical results from the Carathéodory theory of differential equations that the properties on Δ do indeed guarantee well-posedness in this sense. The book [**?**] still is an excellent source for much more refined existence and uniqueness results for nonlinear differential equations under considerably weaker hypotheses.

7.2.2   Definition of (incremental) stability

In general terms, stability of feedback systems is related to whether or not one can impose a bound on the interconnection response in terms of the sizes of the external stimuli. This requires the specification of the external signals and the response signals for a particular stability analysis question of interest.

Concretely, the interconnection (7.2.1) is said to be stable if it is well-posed and if there exist constants $\gamma \geq 0$ and γ_0 such for all $d \in \mathscr{L}_{2e}^l$ with the corresponding response z one has

$$\|z_T\| \leq \gamma \|d_T\| + \gamma_0 \quad \text{for all } T > 0.$$

Similarly incremental stability is characterized with some γ_i by

$$\|(z_1)_T - (z_2)_T\| \leq \gamma_i \|(d_1)_T - (d_2)_T\| \quad \text{for all } T > 0$$

and for all external disturbances $d_1, d_2 \in \mathscr{L}_{2e}^l$ with related responses z_1, z_2. The following formal definition is an equivalent reformulation.

Definition 7.11 The interconnection (7.2.1) shown in Figure 7.1 is *(incrementally) stable* if $I - M\Delta$ has a causal inverse which is (incrementally) bounded.

Although not explicitly included in the definition, the to-be-developed tests for interconnection stability will typically require that M and Δ are themselves causal and (incrementally) bounded. If M, Δ and $(I - M\Delta)^{-1}$ are casual and bounded, the same then holds for $w = \Delta(I - M\Delta)^{-1}(d)$ in (7.2.1). Note, however, that we will have the opportunity to consider different versions of stability properties, for example by confining the external disturbances to some subset $\mathscr{D}$ of $\mathscr{L}_2^l$ or even by dispensing with well-posedness.

Remark 7.12 If M is linear it is important to notice the consequences of our definitions of well-posedness and stability for the interconnection in Figure 7.2 which is described as

$$w = \Delta(z) + d_1 \quad \text{and} \quad z = Mw + d_2. \tag{7.2.4}$$

With $\bar{w} = w - d_1$ the interconnection is seen to be equivalent to

$$\bar{w} = \Delta(z) \quad \text{and} \quad z = M\bar{w} + Md_1 + d_2.$$

If M and Δ are causal, existence and causality of $(I - M\Delta)^{-1}$ then imply that the interconnection (7.2.4) responds with the unique

$$z = (I - M\Delta)^{-1}(Md_1 + d_2) \quad \text{and} \quad w = \Delta(I - M\Delta)^{-1}(Md_1 + d_2) + d_1$$

and with causal dependence to the external disturbances d_1 and d_2. If $(I - M\Delta)^{-1}$ is, in addition, bounded, this interconnection defines a causal and bounded map from $(d_1, d_2) \in \mathscr{L}_{2e}^k \times \mathscr{L}_{2e}^l$ to $(w, z) \in \mathscr{L}_{2e}^k \times \mathscr{L}_{2e}^l$. This is the main reason why we base our definitions of well-posedness and stability on $I - M\Delta$ only.
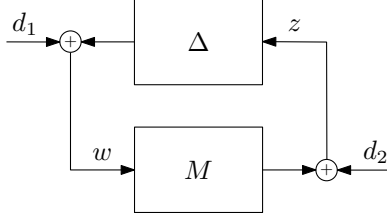
Figure 7.2: Specific feedback interconnection II

In the sequel we will typically assume well-posedness and provide results that allow to (computationally) conclude boundedness of $(I - M\Delta)^{-1}$. The classical small-gain theorem is a simple but eminently useful result that serves this purpose: If (7.2.1) is well-posed, stability of (7.2.1) follows from

$$\|M\|\|\Delta\| < 1. \tag{7.2.5}$$

Although it will be a specialization of more general results discussed later, we nevertheless provide the elementary small-gain arguments in order to illustrate the concepts developed so far. Indeed, since $\|M\Delta\| \leq \|M\|\|\Delta\| < 1$ there exist $\gamma \in (0,1)$ and γ_0 with $\|M\Delta(u_T)\| \leq \gamma\|u_T\| + \gamma_0$ for all $T > 0$ and $u \in \mathscr{L}_{2e}^l$. If $d \in \mathscr{L}_{2e}^l$ and $z = (I - M\Delta)^{-1}(d)$ we infer $z = M\Delta(z) + d$, and due to $1 - \gamma > 0$, we hence conclude for all $T > 0$ that

$$\|z_T\| = \|M\Delta(z)_T + d_T\| \leq \gamma\|z_T\| + \|d_T\| + \gamma_0 \text{ or } \|z_T\| \leq \frac{1}{1-\gamma}\|d_T\| + \frac{\gamma_0}{1-\gamma}.$$

As a matter of motivation for the subsequent development, let us reformulate (7.2.5) in a form that opens the avenue for substantial generalizations. Clearly (7.2.5) holds iff there exists some $\gamma > 0$ such that

$$\|M\| < \gamma \text{ and } \|\Delta\| < \frac{1}{\gamma}.$$

Introduce the abbreviation

$$\Pi_\gamma := \begin{pmatrix} -\gamma I & 0 \\ 0 & \frac{1}{\gamma}I \end{pmatrix}.$$

By Lemma 7.6, a little reflection shows that $\|M\| < \gamma$ implies the existence of m_0 and $\varepsilon > 0$ such that $\frac{1}{\gamma}\|M(w)_T\|^2 - \gamma\|w_T\|^2 \leq -\varepsilon\|w_T\|^2 + m_0$ or

$$\int_0^T \begin{pmatrix} w(t) \\ M(w)(t) \end{pmatrix}^T \Pi_\gamma \begin{pmatrix} w(t) \\ M(w)(t) \end{pmatrix} dt \leq -\varepsilon\|w_T\|^2 + m_0 \text{ for all } T > 0, w \in \mathscr{L}_2^k. \tag{7.2.6}$$

Somewhat more directly, since $\|\Delta\| < 1/\gamma$ there exists δ_0 with $\gamma\|\Delta(z)_T\|^2 \leq \frac{1}{\gamma}\|z_T\|^2 + \delta_0$ or

$$\int_0^T \begin{pmatrix} \Delta(z)(t) \\ z(t) \end{pmatrix}^T \Pi_\gamma \begin{pmatrix} \Delta(z)(t) \\ z(t) \end{pmatrix} dt \geq -\delta_0 \text{ for all } T > 0, z \in \mathscr{L}_2^l. \tag{7.2.7}$$

Let us go one step further and introduce, with any $\Pi = \Pi^T \in \mathbb{R}^{(k+l)\times(k+l)}$, the integral quadratic map

$$\Sigma_\Pi(x) := \int_0^\infty x(t)^T \Pi x(t)\, dt \quad \text{for } x \in \mathscr{L}_2^{k+l}.$$

Then (7.2.6) and (7.2.7) can be respectively expressed as

$$\Sigma_{\Pi_\gamma} \begin{pmatrix} w_T \\ M(w)_T \end{pmatrix} \leq -\varepsilon \|w_T\|^2 + m_0 \quad \text{for all } T > 0,\ w \in \mathscr{L}_2^k \qquad (7.2.8)$$

and

$$\Sigma_{\Pi_\gamma} \begin{pmatrix} \Delta(z)_T \\ z_T \end{pmatrix} \geq -\delta_0 \quad \text{for all } T > 0,\ z \in \mathscr{L}_2^l. \qquad (7.2.9)$$

Let us summarize what we have achieved:

- The small-gain condition (7.2.5) has been re-expressed as (7.2.8) and (7.2.9). This somewhat more involved formulation of the hypotheses allows for extensions of the small-gain theorem, by simply replacing Σ_{Π_γ} with quadratic maps from a far richer class. For example (and as discussed later) if using

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

  rather than Π_γ, the validity of (7.2.8) and (7.2.9) still guarantee that $(I - M\Delta)^{-1}$ is bounded, which is the so-called passivity theorem for stability.

- Note that (7.2.8) imposes an integral quadratic constraint expressed with Σ_{Π_γ} on the truncated signals in the graph of M as defined in the usual fashion by

$$\left\{ \begin{pmatrix} w \\ M(w) \end{pmatrix} :\ w \in \mathscr{L}_2^k \right\}.$$

  This inequality is 'strict' due to the perturbation term involving ε. Similarly (7.2.9) imposes an integral quadratic constraint with the very same Σ_{Π_γ} applied to the truncated signals in the inverse graph of Δ given by

$$\left\{ \begin{pmatrix} \Delta(z) \\ z \end{pmatrix} :\ z \in \mathscr{L}_2^l \right\},$$

  and with the reversed direction of the inequality.

- Suppose that $\delta_0 = m_0 = 0$. For signals in the intersection of these two graphs we infer from (7.2.8) and (7.2.9) that $w = 0$ and hence $z = 0$; geometrically this means that the two graphs only intersect trivially. This provides some flavor of guaranteeing stability in terms of graph separation as thoroughly developed in [?], although we will not rely on this geometric interpretation in this book.

After these motivating remarks we are ready to introduce the class of quadratic maps which will be used in our general theory.

7.2.3 Integral quadratic constraints

With any symmetric matrix $\Pi \in \mathbb{R}^{n \times n}$ we have seen the relevance of the quadratic form

$$\Sigma_\Pi(x) = \langle x, x \rangle_\Pi \ \text{ for } \ x \in \mathscr{L}_2^n$$

that is defined on the basis of the bilinear form

$$\langle u, v \rangle_\Pi = \int_0^\infty u(t)^T \Pi v(t) \, dt \ \text{ for } \ u, v \in \mathscr{L}_2^n.$$

The following definition captures an important generalization.

Definition 7.13 The map $\langle \cdot, \cdot \rangle : \mathscr{L}_2^n \times \mathscr{L}_2^n \to \mathbb{R}$ is *bi-additive* if

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \ \text{ and } \ \langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle \ \text{ for all } \ u, v, w \in \mathscr{L}_2^n.$$

It is bounded if there exists a $c \in \mathbb{R}$ with

$$|\langle u, v \rangle| \leq c \|u\| \|v\| \ \text{ for all } \ u, v \in \mathscr{L}_2^n.$$

If $\Sigma : \mathscr{L}_2^n \to \mathbb{R}$ is defined with such a bi-additive bounded map as

$$\Sigma(x) = \langle x, x \rangle \ \text{ for } \ x \in \mathscr{L}_2^n,$$

we say that Σ is a .

It is easy to check that $\Sigma(u + v) = \Sigma(u) + \langle u, v \rangle + \langle v, u \rangle + \Sigma(v)$ and hence, with boundedness, we arrive at the following to-be-exploited crucial property of Σ:

$$\Sigma(u + v) - \Sigma(u) \leq 2c \|u\| \|v\| + c \|v\|^2 \ \text{ for all } \ u, v \in \mathscr{L}_2^n. \tag{7.2.10}$$

The particular map Σ_Π introduced above does indeed have this property.

Lemma 7.14 Σ_Π *is a bounded quadratic map.*

Proof. Bi-additivity of $\langle \cdot, \cdot \rangle_\Pi$ follows from linearity of integration. Moreover for $u, v \in \mathscr{L}_2^n$ observe that

$$|\langle u, v \rangle_\Pi| = \left| \int_0^\infty u(t)^T \Pi v(t) \, dt \right| \leq \int_0^\infty |u(t)^T \Pi v(t)| \, dt \leq \int_0^\infty \|\Pi\| \|u(t)\| \|v(t)\| \, dt \leq$$

$$\|\Pi\| \sqrt{\int_0^\infty \|u(t)\|^2 \, dt} \sqrt{\int_0^\infty \|v(t)\|^2 \, dt} \leq \|\Pi\| \|u\| \|v\|,$$

with the fourth step being a consequence of the familiar Cauchy-Schwartz inequality for scalar functions. This implies boundedness with the constant $c = \|\Pi\|$. ∎

More generally, for some (measurable) Hermitian-valued bounded function Π defined on the imaginary axis introduce

$$\Sigma_\Pi(x) := \int_{-\infty}^{\infty} \hat{x}(i\omega)^* \Pi(i\omega) \hat{x}(i\omega) \, d\omega \ \text{ for } \ x \in \mathscr{L}_2^n,$$

where the hat indicates taking the Fourier-transform. Σ_Π is a bounded quadratic form since it results from $\langle u, v \rangle_\Pi = \int_{-\infty}^{\infty} \hat{u}(i\omega)^* \Pi(i\omega) \hat{v}(i\omega) \, d\omega$ for $u, v \in \mathscr{L}_2^n$. Indeed $\langle \cdot, \cdot \rangle_\Pi$ is easily seen to be bi-additive and bounded with constant c where $\|\Pi(i\omega)\| \leq c$ for all $\omega \in \mathbb{R} \cup \{\infty\}$. If Π does not depend on frequency, Parseval's theorem implies that the two given definitions in time- and frequency-domain do actually coincide.

We finally introduce the following terminology. If Σ is bounded quadratic we say that

$$\Sigma(x) \leq \sigma_0 \qquad \text{imposes a } \textit{quadratic constraint (QC)} \text{ on the signal } \ x \in \mathscr{L}_2^n \ \text{ and}$$
$$\Sigma(x_T) \leq \sigma_0 \quad \text{ for all } \ T > 0 \ \text{ is a } \textit{hard quadratic constraint} \text{ on } \ x \in \mathscr{L}_2^n.$$

If $\Sigma = \Sigma_\Pi$, these are called **(hard) integral quadratic constraints (IQCs)** on the signal $x \in \mathscr{L}_2^n$ with frequency-dependent multiplier Π.

Example 7.15 Let us reconsider the static diagonal nonlinearity from Example 7.10. With $\phi_\nu : \mathbb{R} \to \mathbb{R}$ define

$$\Phi(x) = \text{col}(\phi_1(x_1), \dots, \phi_k(x_k)) \ \text{ and } \ \Delta(z)(t) := \Phi(z(t)) \ \text{ for } \ t \geq 0, \ z \in \mathscr{L}_{2e}^k.$$

We now illustrate how to derive integral quadratic constraints for this nonlinearity that are useful for stability and performance analysis. For this purpose observe that (7.2.2) persists to hold if we multiply each of them with $\lambda_\nu \geq 0$ and take the sum over $\nu = 1, \dots, k$. With

$$\Pi(\lambda) = \begin{pmatrix} \text{diag}(-\lambda_1, \dots, -\lambda_k) & \text{diag}(\lambda_1 \frac{\alpha_1 + \beta_1}{2}, \dots, \lambda_k \frac{\alpha_k + \beta_k}{2}) \\ \text{diag}(\lambda_1 \frac{\alpha_1 + \beta_1}{2}, \dots, \lambda_k \frac{\alpha_k + \beta_k}{2}) & \text{diag}(-\lambda_1 \alpha_1 \beta_1, \dots, -\lambda_k \alpha_k \beta_k) \end{pmatrix}$$

we get

$$\begin{pmatrix} \Phi(x) \\ x \end{pmatrix}^T \Pi(\lambda) \begin{pmatrix} \Phi(x) \\ x \end{pmatrix} = \sum_{\nu=1}^{k} \lambda_\nu \begin{pmatrix} \phi_\nu(x_\nu) \\ x_\nu \end{pmatrix}^T \begin{pmatrix} -1 & \frac{\alpha_\nu + \beta_\nu}{2} \\ \frac{\alpha_\nu + \beta_\nu}{2} & -\alpha_\nu \beta_\nu \end{pmatrix} \begin{pmatrix} \phi_\nu(x_\nu) \\ x_\nu \end{pmatrix}.$$

We hence conclude that

$$\begin{pmatrix} \Phi(x) \\ x \end{pmatrix}^T \Pi(\lambda) \begin{pmatrix} \Phi(x) \\ x \end{pmatrix} \geq 0 \ \text{ for all } \ x \in \mathbb{R}^k, \ \lambda \in \mathbb{R}^k, \ \lambda \geq 0.$$

This clearly implies

$$\int_0^T \begin{pmatrix} \Phi(z(t)) \\ z(t) \end{pmatrix}^T \Pi(\lambda) \begin{pmatrix} \Phi(z(t)) \\ z(t) \end{pmatrix} \geq 0 \ \text{ for all } \ T > 0, \ z \in \mathscr{L}_2^k, \ \lambda \in \mathbb{R}^k, \ \lambda \geq 0$$

which is nothing but

$$\Sigma_{\Pi(\lambda)} \begin{pmatrix} \Delta(z)_T \\ z_T \end{pmatrix} \geq 0 \ \text{ for all } \ T > 0, \ z \in \mathscr{L}_2^k, \ \lambda \in \mathbb{R}^k, \ \lambda \geq 0.$$

Note that we have actually derived a whole family of hard IQCs with the multipliers $\Pi(\lambda)$ which are linearly parameterized by $\lambda \in \mathbb{R}^k$ that are in turn constrained as $\lambda \geq 0$.

In the case that Δ is defined with functions ϕ_ν which satisfy the incremental sector condition

$$
\begin{pmatrix} \phi_\nu(x_1) - \phi_\nu(x_2) \\ x_1 - x_2 \end{pmatrix}^\top \begin{pmatrix} -1 & \frac{\alpha_\nu + \beta_\nu}{2} \\ \frac{\alpha_\nu + \beta_\nu}{2} & -\alpha_\nu \beta_\nu \end{pmatrix} \begin{pmatrix} \phi_\nu(x_1) - \phi_\nu(x_2) \\ x_1 - x_2 \end{pmatrix} \geq 0 \ \text{ for all } \ x_1, x_2 \in \mathbb{R}
$$

(7.2.11)

and for all $\nu = 1, \ldots, k$, we arrive at the validity of the incremental quadratic constraint

$$
\Sigma_{\Pi(\lambda)} \begin{pmatrix} \Delta(z_1)_T - \Delta(z_1)_T \\ (z_1 - z_2)_T \end{pmatrix} \geq 0 \ \text{ for all } \ T > 0, \ z_1, z_2 \in \mathscr{L}_2^k, \ \lambda \in \mathbb{R}^k, \ \lambda \geq 0.
$$

(7.2.12)

7.3 Hard quadratic constraints

In this section we discuss how to guarantee stability with hard quadratic constraints on the graph of M and the inverse graph of Δ, respectively. We also reveal that incremental quadratic constraints imply incremental stability. Moreover we provide the link to dissipation theory and frequency domain inequalities which render these tests computational.

7.3.1 Stability

Let us come back to the interconnection in Figure 7.1 (see Section 7.2.1) as described by

$$
z = M(w) + d, \ \ w = \Delta(z).
$$

(7.3.1)

Let us now formulate our first fundamental stability result which substantially generalizes the classical small-gain theorem. We allow the external disturbances to be confined to some set $\mathscr{D} \subset \mathscr{L}_{2e}^l$. It is stressed that well-posedness is neither assumed nor concluded. Instead, the conclusion is formulated for all those disturbances $d \in \mathscr{D}$ for which the feedback interconnection does have a response.

Theorem 7.16 *Suppose that $M : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^l$ is causal and bounded, $\Delta : \mathscr{L}_{2e}^l \to \mathscr{L}_{2e}^k$ is causal, Σ is a bounded quadratic form, $\mathscr{D} \subset \mathscr{L}_{2e}^l$, and that*

- *there exist $\varepsilon > 0$ and m_0 such that*

$$
\Sigma \begin{pmatrix} w_T \\ M(w)_T \end{pmatrix} \leq -\varepsilon \|w_T\|^2 + m_0 \ \text{ for all } \ T > 0, \ w \in \mathscr{L}_2^k;
$$

(7.3.2)

- *there exists δ_0 with*

$$
\Sigma \begin{pmatrix} \Delta(z)_T \\ z_T \end{pmatrix} \geq -\delta_0 \ \text{ for all } \ T > 0, \ z \in M(\mathscr{L}_2^k) + \mathscr{D}.
$$

(7.3.3)

Then there exist $\gamma > 0$ and γ_0 such for any $d \in \mathscr{D}$ with response $z \in \mathscr{L}_{2e}^l$ satisfying (7.3.1):

$$\|z_T\|^2 \leq \gamma^2 \|d_T\|^2 + \gamma\gamma_0 \ \text{ for all } \ T > 0. \tag{7.3.4}$$

If M is linear one can choose $\gamma_0 = m_0 + \delta_0$.

Proof. Step 1. Since M is bounded, $\|M(w)_T\|^2 \leq \tilde{\gamma}^2 \|w_T\|^2 + \tilde{\gamma}_0^2$ (for some $\tilde{\gamma}, \tilde{\gamma}_0 \geq 0$) and thus

$$\left\| \begin{pmatrix} w_T \\ M(w)_T \end{pmatrix} \right\| \leq \sqrt{(1+\tilde{\gamma}^2)\|w_T\|^2 + \tilde{\gamma}_0^2} \leq \sqrt{(1+\tilde{\gamma}^2)}\|w_T\| + \tilde{\gamma}_0 \ \text{ for all } \ T > 0, \ w \in \mathscr{L}_{2e}^k.$$

Applying (7.2.10) to $u = \mathrm{col}(w_T, M(w)_T)$ and $v = \mathrm{col}(0, d_T)$ hence leads to

$$\Sigma \begin{pmatrix} w_T \\ M(w)_T + d_T \end{pmatrix} - \Sigma \begin{pmatrix} w_T \\ M(w)_T \end{pmatrix} \leq 2c \left(\sqrt{(1+\tilde{\gamma}^2)}\|w_T\| + \tilde{\gamma}_0 \right) \|d_T\| + c\|d_T\|^2 =$$

$$= \begin{pmatrix} 1 \\ \|w_T\| \\ \|d_T\| \end{pmatrix}^{\top} \begin{pmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \begin{pmatrix} 1 \\ \|w_T\| \\ \|d_T\| \end{pmatrix} \tag{7.3.5}$$

for all $T > 0$ and $(w,d) \in \mathscr{L}_{2e}^k \times \mathscr{L}_{2e}^l$, with σ_{ij} only depending on M and Σ. If M is linear one can choose $\tilde{\gamma}_0 = 0$ which implies $\sigma_{13} = 0$.

Step 2. In this crucial step we show that there exist $\gamma > 0$ and $\widehat{\gamma}_0$ such that

$$\Sigma \begin{pmatrix} w_T \\ M(w)_T + d_T \end{pmatrix} + \frac{1}{\gamma}\|M(w_T)_T + d_T\|^2 - \gamma\|d_T\|^2 \leq \widehat{\gamma}_0 \tag{7.3.6}$$

for all $T > 0$ and $(w,d) \in \mathscr{L}_{2e}^k \times \mathscr{L}_{2e}^l$ as follows. Add (7.3.5) and (7.3.2) to get

$$\Sigma \begin{pmatrix} w_T \\ M(w)_T + d_T \end{pmatrix} \leq \begin{pmatrix} 1 \\ \|w_T\| \\ \|d_T\| \end{pmatrix}^{T} \begin{pmatrix} m_0 & 0 & \sigma_{13} \\ 0 & -\varepsilon & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} 1 \\ \|w_T\| \\ \|d_T\| \end{pmatrix} \tag{7.3.7}$$

for all $T > 0$ and $(w,d) \in \mathscr{L}_{2e}^k \times \mathscr{L}_{2e}^l$. With $\|M(w)_T\| \leq \tilde{\gamma}\|w_T\| + \tilde{\gamma}_0$ and the triangle inequality we infer for all $\gamma > 0$ that

$$\frac{1}{\gamma}\|M(w)_T + d_T\|^2 - \gamma\|d_T\|^2 \leq \frac{1}{\gamma}\left(\tilde{\gamma}\|w_T\| + \tilde{\gamma}_0 + \|d_T\|\right)^2 - \gamma\|d_T\|^2 =$$

$$= \begin{pmatrix} 1 \\ \|w_T\| \\ \|d_T\| \end{pmatrix}^{T} \begin{pmatrix} m_{11}/\gamma & m_{12}/\gamma & m_{13}/\gamma \\ m_{12}/\gamma & m_{22}/\gamma & m_{23}/\gamma \\ m_{13}/\gamma & m_{23}/\gamma & m_{33}/\gamma - \gamma \end{pmatrix} \begin{pmatrix} 1 \\ \|w_T\| \\ \|d_T\| \end{pmatrix} \tag{7.3.8}$$

for all $T > 0$ and $(w,d) \in \mathscr{L}_{2e}^k \times \mathscr{L}_{2e}^l$, where $m_{ij} \in \mathbb{R}$ do not depend upon γ. For any $\widehat{\gamma}_0 > m_0$ observe that there exists some (sufficiently large) $\gamma > 0$ for which

$$\begin{pmatrix} m_0 & 0 & \sigma_{13} \\ 0 & -\varepsilon & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} + \begin{pmatrix} m_{11}/\gamma & m_{12}/\gamma & m_{13}/\gamma \\ m_{12}/\gamma & m_{22}/\gamma & m_{23}/\gamma \\ m_{13}/\gamma & m_{23}/\gamma & m_{33}/\gamma - \gamma \end{pmatrix} \preccurlyeq \begin{pmatrix} \widehat{\gamma}_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.3.9}$$

If we add (7.3.7) and (7.3.8), we can exploit (7.3.9) to arrive at (7.3.6). If M is linear we can choose $\widetilde{\gamma}_0 = 0$ which implies $m_{11} = m_{12} = m_{13} = 0$. We can then take $\widehat{\gamma}_0 = m_0$.

Step 3. To finish the proof choose $d \in \mathscr{D}$ and a corresponding response $z \in \mathscr{L}_{2e}^l$ of (7.3.1). Then $w_T = \Delta(z)_T$ and $z_T = M(w)_T + d_T$. Now observe that the inequality in (7.3.3) holds for $v := M(w_T) + d \in M(\mathscr{L}_2^k) + \mathscr{D}$. On the other hand, by causality we have $v_T = M(w_T)_T + d_T = M(w)_T + d_T = z_T$ and $\Delta(v)_T = \Delta(v_T)_T = \Delta(z_T)_T = \Delta(z)_T = w_T$. This allows to combine (7.3.3) with (7.3.6) in order to infer (7.3.4) for $\gamma_0 = \widehat{\gamma}_0 + \delta_0$ (which equals $m_0 + \delta_0$ if M is linear). ∎

Before giving examples in the next sections let us reflect on the ingredients of this result:

- The proof reveals that the constants γ and γ_0 only depend on M and Σ. In particular, they are independent from Δ which is relevant for robust stability analysis.

- If $d \in \mathscr{D}$ is of finite energy, $d \in \mathscr{L}_2^l$, then (7.3.4) implies that $z \in \mathscr{L}_2^k$ for any response of the feedback system.

- If $I - M\Delta$ has a causal inverse, the conclusion (7.3.4) is valid for all $d \in \mathscr{D}$ and $z = (I - M\Delta)^{-1}(d)$. If, in addition, $\mathscr{D} = \mathscr{L}_2^l$ it translates into $\|(I - M\Delta)^{-1}\|$ being bounded by γ.

- In the practically relevant case $\mathscr{D} = \mathscr{L}_2^l$ we just have $M(\mathscr{L}_2^k) + \mathscr{D} = \mathscr{L}_2^l$ and the quadratic constraint in (7.3.3) has to be verified for all $z \in \mathscr{L}_2^l$.

- In applications we will concentrate on linear M which allows to choose $m_0 = 0$ in (7.3.2) without loss of generality. Indeed let us prove that (7.3.2) for some $m_0 \neq 0$ implies the validity of the same inequality for $m_0 = 0$ as follows. Fix $w \in \mathscr{L}_{2e}^k$ and $T > 0$; for $\nu = 1, 2, \ldots$ we infer from (7.3.2) that

$$\Sigma \begin{pmatrix} (\nu w)_T \\ M(\nu w)_T \end{pmatrix} \leq -\varepsilon \|(\nu w)_T\|^2 + m_0;$$

since M is linear and $\Sigma(\nu x) = \nu^2 \Sigma(x)$ (because Σ is defined by a bi-additive form and ν is a positive integer) we get

$$\Sigma \begin{pmatrix} w_T \\ M(w)_T \end{pmatrix} \leq -\varepsilon \|w_T\|^2 + \frac{m_0}{\nu^2} \overset{\nu \to \infty}{\longrightarrow} -\varepsilon \|w_T\|^2;$$

since $w \in \mathscr{L}_2^k$ and $T > 0$ were arbitrary we arrive at (7.3.2) for $m_0 = 0$.

7.3.2 Small-gain and passivity theorems

Throughout this section we assume that M and Δ are causal and bounded and that $I - M\Delta$ has a causal inverse. Moreover we choose $\mathscr{D} = \mathscr{L}_2^l$.

We have already seen that Theorem 7.16 captures versions of the classical small-gain theorem by specializing Σ to

$$\Sigma_{\Pi_\mu} \begin{pmatrix} w \\ z \end{pmatrix} = \int_0^\infty \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^T \begin{pmatrix} -\mu I & 0 \\ 0 & \frac{1}{\mu} I \end{pmatrix} \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} dt = \int_0^\infty \frac{1}{\mu} z(t)^T z(t) - \gamma w(t)^T w(t) \, dt.$$

Let us formulate and prove one version explicitly.

Corollary 7.17 (Small Gain Theorem) *If* $\|M\|\|\Delta\| < 1$ *then* $(I - M\Delta)^{-1}$ *is bounded.*

Proof. Due to $\|M\|\|\Delta\| < 1$ there exists some $\mu > 0$ with $\|M\| < \mu$ and $\|\Delta\| < \frac{1}{\mu}$. With $\Sigma = \Sigma_{\Pi_\mu}$, the former condition implies that there exists some $\varepsilon > 0$ and m_0 with (7.3.2), while the latter guarantees the existence of δ_0 with (7.3.3). This allows to apply Theorem 7.16 to reach to the conclusion. ∎

Similarly the choice

$$\Sigma \begin{pmatrix} w \\ z \end{pmatrix} = \int_0^\infty \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^T \begin{pmatrix} 0 & \frac{1}{2} I \\ \frac{1}{2} I & 0 \end{pmatrix} \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} dt = \int_0^\infty z(t)^T w(t) \, dt$$

leads to a version of the celebrated passivity theorem. Indeed (7.3.3) just translates into

$$\int_0^T z(t)^T (\Delta z)(t) \, dt \geq \delta_0 \ \text{ for all } \ T > 0, \ z \in \mathscr{L}_2^l$$

which means, in classical terms, that Δ is **passive**. Moreover, (7.3.2) reads as

$$\int_0^T w(t)^T (Mw)(t) \, dt \leq -\varepsilon \int_0^T w(t)^T w(t) \, dt + m_0 \ \text{ for all } \ T > 0, \ w \in \mathscr{L}_2^k$$

which translates into $-M$ being **strictly passive**. We will sometimes say that M is **strictly anti-passive**.

Corollary 7.18 (Passivity Theorem) *If* $-M$ *is strictly passive and* Δ *is passive then* $(I - M\Delta)^{-1}$ *is bounded.*

Note that the positive feedback interconnection with M considered in this chapter can be seen as a negative feedback interconnection with $-M$. The last result can be loosely summarized by saying that a passive system in negative feedback with a strictly passive system leads to a stable interconnection. With ease we have obtained the classical small-gain and passivity theorem which are usually formulated and proved independently. As an example application of the passivity theorem we will recall in Section 7.4.4 the standard proof of the Popov criterion.

7.3.3   LTI systems in the forward path

The power of the developed framework for actual computations can be particularly well-illustrated for the situation that M is described by an LTI system. More specifically let us consider the interconnection

$$\begin{cases} \dot{x} = Ax + Bw, \qquad x(0) = x_0 \\ w = \Delta(z) + d_1 \\ z = Cx + Dw + d_2 \end{cases} \tag{7.3.10}$$

with A being Hurwitz. For any trajectory of (7.3.10) the variation-of-constants formula

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} Bw(\tau) \mathrm{d}\tau \ \text{ for } \ t \geq 0$$

reveals that $x(t) = \bar{x}(t) + d_0(t)$ where $d_0(t) = e^{At}x_0$ and $\bar{x}$ is the state-response of the interconnection

$$\begin{cases} \dot{\bar{x}} = A\bar{x} + Bw, \qquad\qquad \bar{x}(0) = 0 \\ w = \Delta(z) + d_1 \\ z = C\bar{x} + Dw + (Cd_0 + d_2) \end{cases} \tag{7.3.11}$$

with zero initial condition. To fix notations suppose that

$$y = Mu \quad \text{ is described by } \begin{cases} \dot{\bar{x}} = A\bar{x} + Bu \qquad \bar{x}(0) = 0 \\ y = C\bar{x} + Du \end{cases}. \tag{7.3.12}$$

Then (7.3.11) is exactly the interconnection in Figure 7.2 with d_2 replaced by $Cd_0 + d_2$. In view of Remark 7.12 the interconnection (7.3.10) hence translates with

$$\bar{x} = x - d_0, \ \ \bar{w} = w - d_1 \ \text{ and } \ d_0(t) = e^{At}x_0, \ \ d = Cd_0 + Md_1 + d_2$$

into the standard configuration of Figure 7.1 as described by

$$\begin{cases} \dot{\bar{x}} = A\bar{x} + B\bar{w}, \qquad \bar{x}(0) = 0 \\ \bar{w} = \Delta(z) \\ z = C\bar{x} + D\bar{w} + d \end{cases} \tag{7.3.13}$$

Clearly well-posedness of (7.3.13) is equivalent to the fact that (7.3.10) has for all x_0, d_1, d_2 unique responses w and z that depend causally on d_1 and d_2. If well-posed, let us now analyze the consequence of (7.3.13) being stable. For this purpose note that

$$\|d_0\| \leq \sqrt{\int_0^\infty \|e^{At}x_0\|^2 \mathrm{d}t} \leq \gamma_A \|x_0\| \ \text{ with } \ \gamma_A := \sqrt{\int_0^\infty \|e^{At}\|^2 \mathrm{d}t} \tag{7.3.14}$$

where we use the spectral matrix norm in the definition of γ_A. Below we will also exploit that the state-responses of $\dot{x} = Ax + Bw$, $x(0) = x_0$, can be estimated for all $w \in \mathscr{L}_2^k$ as

$$\|x\| \leq \gamma_{AB}\|w\| + \gamma_A\|x_0\| \ \text{ with } \ \gamma_{AB} := \|(sI - A)^{-1}B\|_\infty. \tag{7.3.15}$$

If (7.3.13) is shown to be stable there exists constants $\gamma \geq 0$, γ_0 such that $\|z_T\| \leq \gamma \|d_T\| + \gamma_0 = \gamma \|(Cd_0)_T + (Md_1)_T + (d_2)_T\| + \gamma_0$ or

$$\|z_T\| \leq (\gamma \gamma_A \|C\|) \|x_0\| + (\gamma \|M\|) \|(d_1)_T\| + \gamma \|(d_2)_T\| + \gamma_0 \quad \text{for all } T > 0. \qquad (7.3.16)$$

This very same inequality also holds for the trajectories of (7.3.10); it is actually compatible with our abstract stability definition if viewing both d and x_0 as inputs to the interconnection. In the case that $d_1 \in \mathscr{L}_2^k$ and $d_2 \in \mathscr{L}_2^l$ it is particularly relevant to observe that (7.3.16) implies $z \in \mathscr{L}_2^l$ and thus $w = \Delta(z) + d_1 \in \mathscr{L}_2^k$ and hence $x \in \mathscr{L}_2^n$ as well as $\dot{x} \in \mathscr{L}_2^n$ due to (7.3.10), which in turn leads to

$$\lim_{t \to \infty} x(t) = 0.$$

Moreover (7.3.16) implies

$$\|z\| \leq (\gamma \gamma_A \|C\|) \|x_0\| + (\gamma \|M\|) \|d_1\| + \gamma \|d_2\| + \gamma_0.$$

Due to (7.3.15) and $\|w\| \leq \|\Delta\| \|z\| + \|d_1\|$, we infer the existence of constants $\tilde{\gamma}$, γ_0, γ_1, γ_2 (that can be expressed in terms of γ, $\|\Delta\|$ and $\|M\|$, γ_A, γ_{AB}) such that

$$\|x\| \leq \tilde{\gamma} \|x_0\| + \gamma_1 \|d_1\| + \gamma_2 \|d_2\| + \gamma_0.$$

Therefore input-output stability of (7.3.13) makes it possible to conclude all these input-to-output and input-to-state stability properties for (7.3.10), irrespective of the nature of Δ and independently from the technique how stability of (7.3.13) has been verified.

In summary, in this section we have demonstrated how the stability analysis of (7.3.10) with non-zero initial conditions and two external inputs can be generically reduced to analyzing the stability of (7.3.13) with zero initial conditions in which the external disturbance d is possibly restricted to some domain of definition $\mathscr{D}$.

7.3.4 Circle criteria

For a concrete stability analysis problem let us come back to Example 7.10, the interconnection of a strictly proper stable LTI system (defining a causal and bounded linear M for $x_0 = 0$) and a diagonal sector bounded nonlinearly (defining the causal and bounded Δ) which had been shown to be well-posed. In Example 7.15 we have obtained a whole family of valid hard IQCs with $\Sigma_{\Pi(\lambda)}$ for all $\lambda \in \mathbb{R}^k$ satisfying $\lambda \geq 0$ and for any such nonlinearity.

Let us now emphasize the key element which renders all the stability tests in this chapter computational. For this purpose we only need to recall that (7.3.2) is related to a strict dissipativity constraint. With the system's transfer matrix $T(s) = C(sI - A)^{-1}B$ we have proved that (7.3.2) is equivalent to the frequency domain inequality

$$\begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \Pi(\lambda) \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} \prec 0 \quad \text{for all } \omega \in \mathbb{R} \cup \{\infty\} \qquad (7.3.17)$$

Figure 7.3: Saturation nonlinearity in feedback interconnection

or, due to the KYP-lemma, to the existence of some X with

$$\begin{pmatrix} A^T X + XA & XB \\ B^T X & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^T \Pi(\lambda) \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \prec 0. \tag{7.3.18}$$

This leads to the following version of a multi-loop circle criterion.

Corollary 7.19 (Circle Criterion) *Suppose there exists some $\lambda \in \mathbb{R}^k$ with $\lambda \geq 0$ and $X = X^T$ satisfying (7.3.18). Then the system (7.2.3) defining $d \to z$ has finite $\mathcal{L}_2$-gain.*

Proof. Clearly (7.3.3) is satisfied with $\delta_0 = 0$ for $\Sigma = \Sigma_{\Pi(\lambda)}$ and all λ. If (7.3.18) is feasible for some particular $\lambda_* \geq 0$, we infer that (7.3.2) is valid for $\Sigma = \Sigma_{\Pi(\lambda_*)}$ and some $\varepsilon > 0$ as well as $m_0 = 0$. By Theorem 7.16 we infer that $(I - M\Delta)^{-1}$ which is described by (7.2.3) is bounded. ∎

Let us briefly touch upon the relation to the classical circle criterion. If $k = 1$ then T is a SISO transfer function and (7.3.17) translates with $\alpha = \alpha_1$ and $\beta = \beta_1$ into

$$\begin{pmatrix} 1 \\ T(i\omega) \end{pmatrix}^* \begin{pmatrix} -1 & \frac{\alpha+\beta}{2} \\ \frac{\alpha+\beta}{2} & -\alpha\beta \end{pmatrix} \begin{pmatrix} 1 \\ T(i\omega) \end{pmatrix} \prec 0 \text{ for all } \omega \in \mathbb{R} \cup \{\infty\}.$$

This just means that $\{T(i\omega) : \omega \in \mathbb{R} \cup \{\infty\}\}$ is contained in a circle in the complex plane that is defined by (α, β), and which might degenerate to a half plane. For example if $\alpha = 0$ we obtain the well-known condition

$$\text{Re}(T(i\omega)) < \frac{1}{\beta} \text{ for all } \omega \in \mathbb{R} \cup \{\infty\}$$

on the Nyquist curve of T for interconnection stability. Those readers acquainted with the classical results might realize the ease with which we obtained the multi-loop extension without referring to any technical tricks such as loop transformations.

From this example we can extract why we have actually obtained a robust stability result. Indeed, we have specified a whole class of nonlinear maps Δ described in terms of the sector parameters (α_ν, β_ν) for $\nu = 1, \ldots, k$. This enabled us to determine a family of quadratic constraints that are valid for the inverse graph of all these nonlinearities Δ. If any of the quadratic constraints for the graph of the system M is also satisfied, we have proved stability of the feedback interconnection. For LTI systems M the latter condition translates into the FDI (7.3.18) or, in turn, into feasibility of the LMI (7.3.18). Since the multipliers $\Pi(\lambda)$ are nicely parameterized (because λ enters (7.3.18) affinely and is only subject to the LP constraint $\lambda \geq 0$), the search for a suitable multiplier to render the quadratic constraint for M satisfied just translates into an LMI problem. This is a particular version of the generic procedure of robust stability and performance analysis with (integral) quadratic constraints.

Example 7.20 For the family of transfer functions

$$M(s) = \frac{(a-s)s}{s^3 + s^2 + 2s + 1}, \quad a \in [0.2, 2]$$

a	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8
b_{cir}	2.19	1.6	1.24	1	0.83	0.705	0.611	0.537
b_{par}	4.55	2.91	2.1	1.62	1.31	1.09	0.925	0.803

Table 7.1: Computed stability margins in Example 7.20.

and the saturation

$$\phi(x) = \begin{cases} bx & \text{for} \quad |x| \leq 1 \\ b\,\text{sign}(x) & \text{for} \quad |x| > 1 \end{cases}$$

with $b > 0$, consider the interconnection in Figure 7.3. We are interested in finding the largest value of b for which one can prove stability of the interconnection. Clearly ϕ satisfies a sector condition with constants $\alpha = 0$ and $\beta = b$. The largest b for which the circle criterion guarantees stability equals the largest b for which there exists some $\lambda \geq 0$ such that the LMI (7.3.18) with

$$\Pi(\lambda) = \begin{pmatrix} -\lambda & \lambda \frac{b}{2} \\ \lambda \frac{b}{2} & 0 \end{pmatrix}$$

is feasible. Since the LMI is strict, we can restrict the search to $\lambda > 0$. Moreover by homogeneity we can divide by λ and $b/2$ and check feasibility of the very same LMI with $\Pi(\lambda)$ substituted by

$$\begin{pmatrix} \mu & 1 \\ 1 & 0 \end{pmatrix} \quad \text{with} \quad \mu := -\frac{2}{b}.$$

Since μ now enters the LMI linearly, it is possible to directly maximize μ over the resulting LMI constraint to obtain μ_*. In this fashion one computes with $b_{\text{cir}} = -2/\mu_*$ the largest possible b for which stability can be assured by the circle criterion. Note that it is easy to also determine b_{par}, the largest $b > 0$ such that the interconnection is stable for all **linear** functions $\phi(x) = \delta x$ with $\delta \in [0, b]$. Numerical results are given in Table 7.1. They confirm the obvious inequality $b_{\text{cir}} \leq b_{\text{par}}$ while the quality of the bound b_{cir} remains unclear. We will indeed confirm by improved tests that the interconnection remains stable for saturation parameters $b > b_{\text{cir}}$.

At this point we can nicely illustrate the generic idea of how to reduce conservatism of such stability analysis results for multivariable nonlinearities. So far we have constructed diagonal multipliers which share their structure with the nonlinearity. It is rather pleasing that indirectly described full block multipliers can be employed as well, with the benefit of reducing conservatism at the cost of higher computational complexity. Indeed let us introduce the abbreviation $\Delta(\delta) = \text{diag}(\delta_1, \ldots, \delta_k)$ and consider the set of multipliers

$$\boldsymbol{\Pi} = \left\{ \Pi \in \mathbb{R}^{2k \times 2k} : \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}^T \Pi \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} \succcurlyeq 0, \ \Pi_{\nu\nu} \leq 0 \text{ for } \delta_\nu \in \{\alpha_\nu, \beta_\nu\}, \ \nu = 1, \ldots, k \right\}$$

(7.3.19)

which is LMIable and which we encountered earlier for parametric uncertainties.

Lemma 7.21 *If $\Pi \in \mathbf{\Pi}$ then*

$$\left(\begin{array}{c} \Phi(x) \\ x \end{array} \right)^T \Pi \left(\begin{array}{c} \Phi(x) \\ x \end{array} \right) \geq 0 \ \textit{for all} \ x \in \mathbb{R}^k.$$

Proof. Choose any $x \in \mathbb{R}^k$. Let us define

$$\delta_v := \left\{ \begin{array}{ll} \frac{\phi_v(x_v)}{x_v} \in [\alpha_v, \beta_v] & \text{if} \quad x_v \neq 0, \\ \text{any } \delta_v \in [\alpha_v, \beta_v] & \text{if} \quad x_v = 0. \end{array} \right.$$

We then infer that $\delta_v x_v = \phi_v(x_v)$ for **all** $v = 1, \ldots, k$ which in turn implies

$$\left(\begin{array}{c} \Delta(\delta) \\ I \end{array} \right) x = \left(\begin{array}{c} \Delta(\delta)x \\ x \end{array} \right) = \left(\begin{array}{c} \Phi(x) \\ x \end{array} \right).$$

Now recall from the discussion of parametric uncertainties that $\Pi \in \mathbf{\Pi}$ implies

$$\left(\begin{array}{c} \Delta(\delta) \\ I \end{array} \right)^\top \Pi \left(\begin{array}{c} \Delta(\delta) \\ I \end{array} \right) \succcurlyeq 0 \ \text{ and hence } \ \left(\begin{array}{c} \Delta(\delta)x \\ x \end{array} \right)^\top \Pi \left(\begin{array}{c} \Delta(\delta)x \\ x \end{array} \right) \geq 0.$$

■

Literally as argued in Example 7.15 we conclude that Σ_Π for $\Pi \in \mathbf{\Pi}$ satisfies (7.3.3) for all sector bounded nonlinearities under consideration. This proves the following less conservative version of the multi-loop circle criterion.

Corollary 7.22 (Full Block Multiplier Circle Criterion) *Suppose there exists some $\Pi \in \mathbf{\Pi}$ and $X = X^T$ with*

$$\left(\begin{array}{cc} A^\top X + XA & XB \\ B^\top X & 0 \end{array} \right) + \left(\begin{array}{cc} 0 & I \\ C & D \end{array} \right)^\top \Pi \left(\begin{array}{cc} 0 & I \\ C & D \end{array} \right) \prec 0. \tag{7.3.20}$$

Then the system (7.2.3) *defining $d \rightarrow z$ has finite $\mathcal{L}_2$-gain.*

These ideas point towards a multitude of extensions which require only minor modifications of the arguments. Here is a selection of some possibilities:

- All given results remain valid for sector bounded nonlinearities that explicitly depend on time (with technical properties only dictated by guaranteeing well-posedness).

- Full block multiplier stability results are easy to derive for possibly non-diagonal nonlinearities $\Phi : \mathbb{R}^k \times [0, \infty) \rightarrow \mathbb{R}^k$ for which

$$\left\{ \mathrm{diag}\left(\frac{\Phi_1(x,t)}{x_1}, \ldots, \frac{\Phi_k(x,t)}{x_k} \right) : \ (x,t) \in \mathbb{R}^k \times [0, \infty), \ x_v \neq 0, \ v = 1, \ldots, k \right\}$$

is contained in some compact polytope of $\mathbb{R}^{k \times k}$.

- For systems with k inputs and l outputs one can even further expand to nonlinear feedbacks $\Phi : \mathbb{R}^l \times [0, \infty) \to \mathbb{R}^k$ that satisfy the multivariable sector constraints

$$(\Phi(x,t) - K_\nu x)^T (L_\nu x - \Phi(x,t)) \geq 0 \ \text{ for all } \ (x,t) \in \mathbb{R}^l \times [0, \infty), \ \nu = 1, \dots, N$$

  with $K_\nu \in \mathbb{R}^{l \times k}$ and $L_\nu \in \mathbb{R}^{l \times k}$. Stability is assured with the LMI class of multipliers

$$\text{conv} \left\{ \begin{pmatrix} -2I & L_\nu + K_\nu \\ L_\nu^T + K_\nu^T & -K_\nu^T L_\nu - L_\nu^T K_\nu \end{pmatrix} : \ \nu = 1, \dots, N \right\}.$$

- The feedback loop can be defined in terms of multi-valued nonlinearities, with (7.3.3) replaced by a quadratic constraint on the correspondingly defined inverse graph. The investigation of well-posedness then requires extra attention.

Let us summarize our findings if M is an LTI system with realization given by (A, B, C, D) in which A is stable. In order to guarantee stability of the feedback interconnection (7.3.1), the main effort was devoted to finding a suitable class Σ's for which (7.3.3) is satisfied. If this class is parameterized as Σ_Π with $\Pi = \Pi^T \in \mathbb{R}^{(k+l) \times (k+l)}$ constrained by LMIs, stability is guaranteed by feasibility of the dissipation LMI (7.3.20) subject to the respective constraints on Π. This generic construction of stability tests will be shown to systematically extend to multipliers that involve dynamics which often results in considerably reduced conservatism.

7.3.5 Incremental stability

For the feedback interconnection (7.3.1) let us now derive a criterion for incremental stability. This requires to impose hard quadratic constraints on the incremental graph of M and the inverse incremental graph of Δ as made precise in the following result.

Theorem 7.23 *Let $M : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^l$ be causal and incrementally bounded, $\Delta : \mathscr{L}_{2e}^l \to \mathscr{L}_{2e}^k$ be causal, Σ be a bounded quadratic form and $\mathscr{D} \subset \mathscr{L}_{2e}^l$. Suppose that there exists $\varepsilon > 0$ with*

$$\Sigma \begin{pmatrix} (w_1 - w_2)_T \\ M(w_1)_T - M(w_2)_T \end{pmatrix} \leq -\varepsilon \|(w_1 - w_2)_T\|^2 \ \text{ for all } \ T > 0, \ w_1, w_2 \in \mathscr{L}_2^k \tag{7.3.21}$$

and that

$$\Sigma \begin{pmatrix} \Delta(z_1)_T - \Delta(z_2)_T \\ (z_1 - z_2)_T \end{pmatrix} \geq 0 \ \text{ for all } \ T > 0, \ z_1, z_2 \in M(\mathscr{L}_2^k) + \mathscr{D}. \tag{7.3.22}$$

Then there exists some $\gamma > 0$ (only depending on M and Σ) such that $d_1, d_2 \in \mathscr{D}$ implies for all corresponding responses z_1, z_2 of the feedback interconnection (7.3.1) *that*

$$\|(z_1 - z_2)_T\| \leq \gamma \|(d_1 - d_2)_T\| \ \text{ for all } \ T > 0. \tag{7.3.23}$$

Proof. The proof is almost identical to that of Theorem 7.16.

Step 1. Since M is incrementally bounded there exists some $\tilde{\gamma} \geq 0$ with

$$\left\| \begin{pmatrix} (w_1 - w_1)_T \\ M(w_1)_T - M(w_2)_T \end{pmatrix} \right\| \leq \sqrt{1 + \tilde{\gamma}^2} \, \|(w_1 - w_2)_T\| \quad \text{for all } T > 0, \ w_1, w_2 \in \mathscr{L}_{2e}^k.$$

Using (7.2.10) for $u = \text{col}((w_1 - w_2)_T, M(w_1)_T - M(w_2)_T)$ and $v = \text{col}(0, (d_1 - d_2)_T)$ shows

$$\Sigma \begin{pmatrix} (w_1 - w_2)_T \\ M(w_1)_T - M(w_2)_T + (d_1 - d_2)_T \end{pmatrix} - \Sigma \begin{pmatrix} (w_1 - w_2)_T \\ M(w_1)_T - M(w_2)_T \end{pmatrix} \leq$$

$$\leq 2c \left(\sqrt{1 + \tilde{\gamma}^2} \, \|(w_1 - w_2)_T\| \right) \|(d_1 - d_2)_T\| + c\|(d_1 - d_2)_T\|^2 =$$

$$= \begin{pmatrix} \|(w_1 - w_2)_T\| \\ \|(d_1 - d_2)_T\| \end{pmatrix}^T \begin{pmatrix} 0 & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \|(w_1 - w_2)_T\| \\ \|(d_1 - d_2)_T\| \end{pmatrix} \quad (7.3.24)$$

for all $T > 0$ and $(w_1, d_1), (w_2, d_2) \in \mathscr{L}_{2e}^k \times \mathscr{L}_{2e}^l$, with σ_{12}, σ_{22} only depending on M and Σ.

Step 2. Again with incremental boundedness of M we infer for all $\gamma > 0$ that

$$\frac{1}{\gamma} \|M(w_1)_T - M(w_2)_T + (d_1 - d_2)_T\|^2 - \gamma \|(d_1 - d_2)_T\|^2 \leq$$

$$\leq \frac{1}{\gamma} \left(\tilde{\gamma} \|(w_1 - w_2)_T\| + \|(d_1 - d_2)_T\| \right)^2 - \gamma \|(d_1 - d_2)_T\|^2 =$$

$$= \begin{pmatrix} \|(w_1 - w_2)_T\| \\ \|(d_1 - d_2)_T\| \end{pmatrix}^T \begin{pmatrix} m_{11}/\gamma & m_{12}/\gamma \\ m_{12}/\gamma & m_{22}/\gamma - \gamma \end{pmatrix} \begin{pmatrix} \|(w_1 - w_2)_T\| \\ \|(d_1 - d_2)_T\| \end{pmatrix} \quad (7.3.25)$$

for all $T > 0$ and $(w_1, d_1), (w_2, d_2) \in \mathscr{L}_{2e}^k \times \mathscr{L}_{2e}^l$, with m_{ij} independent from γ. Now choose $\gamma > 0$ so large that

$$\begin{pmatrix} -\varepsilon & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} + \begin{pmatrix} m_{11}/\gamma & m_{12}/\gamma \\ m_{21}/\gamma & m_{22}/\gamma - \gamma \end{pmatrix} \preccurlyeq 0.$$

If we then add (7.3.24) with (7.3.21) and (7.3.25) we arrive at

$$\Sigma \begin{pmatrix} (w_1 - w_2)_T \\ M(w_1)_T - M(w_2)_T + (d_1 - d_2)_T \end{pmatrix} +$$

$$+ \frac{1}{\gamma} \|M(w_1)_T - M(w_2)_T + (d_1 - d_2)_T\|^2 - \gamma \|(d_1 - d_2)_T\|^2 \leq 0 \quad (7.3.26)$$

for all $T > 0$ and $(w_1, d_1), (w_2, d_2) \in \mathscr{L}_{2e}^k \times \mathscr{L}_{2e}^l$.

Step 3. To finish the proof choose $d_j \in \mathscr{D}$ and some corresponding responses $z_j \in \mathscr{L}_{2e}^l$ of (7.3.1) for $j = 1, 2$. Then $(w_j)_T = \Delta(z_j)_T$ and $(z_j)_T = M(w_j)_T + (d_j)_T$. Now observe that the inequality in (7.3.22) holds for $v_j := M((w_j)_T) + d_j \in M(\mathscr{L}_2^k) + \mathscr{D}$. On the other hand we have $(v_j)_T = M((w_j)_T)_T + (d_j)_T = M(w_j)_T + (d_j)_T = (z_j)_T$ and hence $\Delta(v_j)_T = \Delta((v_j)_T)_T = \Delta((z_j)_T)_T = \Delta(z_j)_T = (w_j)_T$. This allows to combine (7.3.22) with (7.3.26) in order to infer (7.3.23). ∎

Consequently, if d_1 and d_2 are disturbances for which $d_1 - d_2$ is of finite energy, any two responses z_1 and z_2 also have this property. Moreover if d_1 and d_2 are close to each other ($\|d_1 - d_2\|$ is small),

the same holds for z_1 and z_2 ($\|z_1 - z_2\|$ is small as well). This important continuity property is the most relevant benefit of incremental stability results at this point.

Again, if $I - M\Delta$ has a causal inverse, the conclusion (7.3.4) is valid for all $d \in \mathscr{D}$, and in case of $\mathscr{D} = \mathscr{L}_2^l$ it translates into $\|(I - M\Delta)^{-1}\|_i \leq \gamma$. If, in addition, $M(0) = 0$ and $\Delta(0) = 0$, one easily checks that $(I - M\Delta)^{-1}(0) = 0$ and one can as well conclude $\|(I - M\Delta)^{-1}\| \leq \gamma$.

Let us look back to the circle criterion in Corollary 7.22. If the nonlinearities of the feedback interconnection in Example 7.10 do satisfy an incremental sector condition, we have seen that the incremental IQC (7.2.12) is valid. Since M is linear, (7.3.2) for $m_0 = 0$ is identical to (7.3.21). Therefore feasibility of the very same LMI (7.3.18) implies incremental stability of the system $d \to z$ defined by (7.2.3) with $x_0 = 0$.

7.4 Soft quadratic constraints

In the interconnection of Figure 7.1 (see Section 7.2.1) we now assume that M is linear. In contrast to the previous section, we intend to prove stability by just imposing the soft versions of quadratic constraints on un-truncated $\mathscr{L}_2$ signals. We have to pay the price that well-posedness as well as the constraint on the possibly nonlinear system have to be checked for all $\tau\Delta$ with $\tau \in [0,1]$. This leads to the benefit of stronger stability characterizations as will be seen in examples.

7.4.1 Simple version of main QC stability result

Theorem 7.24 *Let $M : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^l$ and $\Delta : \mathscr{L}_{2e}^l \to \mathscr{L}_{2e}^k$ be causal and bounded, M be linear and Σ be a bounded quadratic form. Suppose that*

- *$I - \tau M\Delta$ has a causal inverse for each $\tau \in [0,1]$;*

- *there exists $\varepsilon > 0$ such that*

$$\Sigma \begin{pmatrix} w \\ Mw \end{pmatrix} \leq -\varepsilon \|w\|^2 \ \text{for all} \ w \in \mathscr{L}_2^k; \tag{7.4.1}$$

- *there exists δ_0 such that*

$$\Sigma \begin{pmatrix} \tau\Delta(z) \\ z \end{pmatrix} \geq -\delta_0 \ \text{for all} \ \tau \in [0,1], z \in \mathscr{L}_2^l. \tag{7.4.2}$$

Then $(I - M\Delta)^{-1}$ is bounded.

Proof. For $\tau \in [0,1]$ let us introduce the abbreviation $S_\tau = (I - \tau M\Delta)^{-1}$.

Step 1. Only in proving the following key fact we make use of the quadratic constraints: There exists a (τ-independent) $\gamma > 0$ such that

$$\tau \in [0,1] \text{ and } S_\tau(\mathscr{L}_2^k) \subset \mathscr{L}_2^l \text{ imply } \|S_\tau\| \leq \gamma. \qquad (7.4.3)$$

In other words, whenever S_τ maps $\mathscr{L}_2^k$ into signals of finite energy, the quadratic constraints just guarantee the existence of a bound on $\|S_\tau\|$ that is uniform in τ.

As in Step 1 of the proof of Theorem 7.16 one shows that there exist $\sigma_{12}, \sigma_{22} \in \mathbb{R}$ with

$$\Sigma \left(\begin{array}{c} w \\ Mw+d \end{array} \right) - \Sigma \left(\begin{array}{c} w \\ Mw \end{array} \right) \leq \left(\begin{array}{c} \|w\| \\ \|d\| \end{array} \right)^T \left(\begin{array}{cc} 0 & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{array} \right) \left(\begin{array}{c} \|w\| \\ \|d\| \end{array} \right) \qquad (7.4.4)$$

for all $w \in \mathscr{L}_2^k$, $d \in \mathscr{L}_2^l$. Similarly, for any $\gamma > 0$ we have

$$\frac{1}{\gamma}\|Mw+d\|^2 - \gamma\|d\|^2 \leq \frac{1}{\gamma}(\|M\|\|w\| + \|d\|)^2 - \gamma\|d\|^2 =$$

$$= \left(\begin{array}{c} \|w\| \\ \|d\| \end{array} \right)^T \left(\begin{array}{cc} m_{11}/\gamma & m_{12}/\gamma \\ m_{12}/\gamma & m_{22}/\gamma - \gamma \end{array} \right) \left(\begin{array}{c} \|w\| \\ \|d\| \end{array} \right) \text{ for all } w \in \mathscr{L}_2^k, d \in \mathscr{L}_2^l, \quad (7.4.5)$$

where m_{ij} only depend on $\|M\|$. If we add (7.4.1), (7.4.4) and (7.4.5) we infer as in Step 1 of the proof of Theorem 7.16 that there exists some $\gamma > 0$ with

$$\Sigma \left(\begin{array}{c} w \\ Mw+d \end{array} \right) + \frac{1}{\gamma}\|Mw+d\|^2 - \gamma\|d\|^2 \leq 0 \text{ for all } w \in \mathscr{L}_2^k, d \in \mathscr{L}_2^l. \qquad (7.4.6)$$

Now fix $d \in \mathscr{L}_2^l$. We can use the hypothesis in (7.4.3) to infer $z := S_\tau(d) \in \mathscr{L}_2^l$ and hence also $w := \tau\Delta(z) \in \mathscr{L}_2^k$. Since $z = Mw + d$ we can combine (7.4.6) with (7.4.2) in order to arrive at

$$\frac{1}{\gamma}\|S_\tau(d)\|^2 - \gamma\|d\|^2 = \frac{1}{\gamma}\|Mw+d\|^2 - \gamma\|d\|^2 \leq \delta_0.$$

Since $d \in \mathscr{L}_2^l$ was arbitrary we end up with $\|S_\tau\| \leq \gamma$.

Step 2. Choose any $\rho_0 > 0$ which satisfies $\rho_0 < 1/(\|M\Delta\|\gamma)$. Then

$$\tau \in [0,1], \ \tau + \rho \in [0,1], \ |\rho| \leq \rho_0, \ S_\tau(\mathscr{L}_2^l) \subset \mathscr{L}_2^l \text{ imply } S_{\tau+\rho}(\mathscr{L}_2^l) \subset \mathscr{L}_2^l. \qquad (7.4.7)$$

For any ρ, τ as in (7.4.7) note that $\|S_\tau\| \leq \gamma$ by (7.4.3). Moreover if $d \in \mathscr{L}_2^l$, observe that $z = S_{\tau+\rho}(d)$ iff $z - \tau M\Delta(z) - \rho M\Delta(z) = d$ iff $z - \tau M\Delta(z) = \rho M\Delta(z) + d$ iff

$$v = \rho M\Delta(z) + d, \ z = S_\tau(v),$$

as illustrated in Figure 7.4. We are now in the position to just apply the small-gain theorem: Since $\|\rho M\Delta\|\|S_\tau\| \leq \rho_0\|M\Delta\|\gamma < 1$ we conclude $v \in \mathscr{L}_2^l$ and hence $z = S_\tau(v) \in \mathscr{L}_2^l$ and thus in turn also $S_{\tau+\rho}(d) = z \in \mathscr{L}_2^k$.
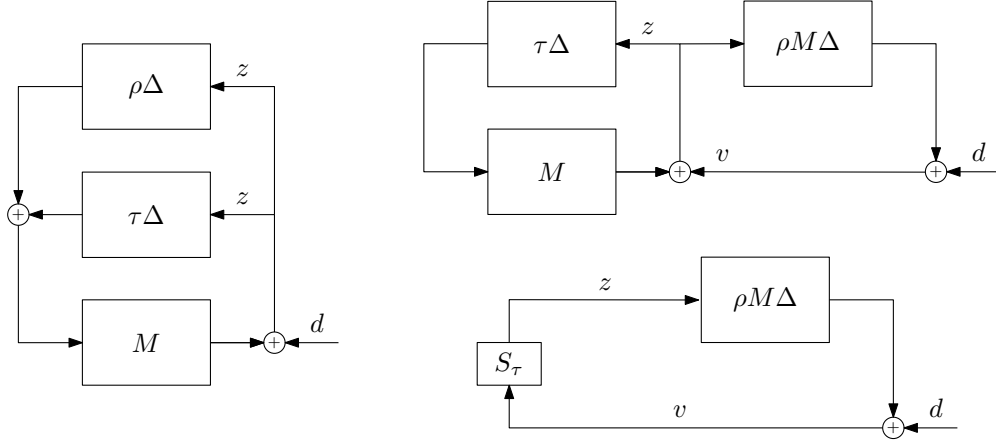
Figure 7.4: Interconnection in proof of Theorem 7.35.

Step 3. Clearly $S_0(\mathscr{L}_2^l) \subset \mathscr{L}_2^l$. Since ρ_0 in Step 2 does not depend on τ, we can inductively apply (7.4.7) in order to infer $S_\tau(\mathscr{L}_2^l) \subset \mathscr{L}_2^l$ for $\tau \in [0, \nu\rho_0] \cap [0,1]$ and all $\nu = 1, 2, \ldots$, and thus in particular for $\tau = 1$. Then (7.4.3) implies that $S_1 = (I - M\Delta)^{-1}$ is bounded.   ∎

The proof actually reveals that there exist some γ only depending on M and Σ such that

$$\|(I - M\Delta)^{-1}(d)\|^2 \leq \gamma^2 \|d\| + \gamma\delta_0 \quad \text{for all } d \in \mathscr{L}_2^l.$$

The fact that γ does not depend on Δ is useful for checking robustness of stability.

Remark 7.25 For any fixed $z \in \mathscr{L}_2^2$ it is often not difficult to analyze the dependence of

$$\Sigma \begin{pmatrix} \tau\Delta(z) \\ z \end{pmatrix} \tag{7.4.8}$$

on the parameter τ. For bounded quadratic Σ's it can be shown that this is just a polynomial of degree two in τ. If the dependence of (7.4.8) is concave then (7.4.2) only needs to be verified for $\tau = 0$ and $\tau = 1$.

7.4.2 Zames-Falb multipliers

This section serves to illustrate the advantage of soft IQCs versus hard IQCs by developing the theory around the classical and celebrated Zames-Falb multipliers. For this purpose let us consider

$$\dot{x} = Ax + B\phi(Cx + d), \quad x(0) = 0 \tag{7.4.9}$$

with A being Hurwitz and with $\phi : \mathbb{R} \to \mathbb{R}$. Recall from Section 7.3.3 that well-posedness and stability of this standard feedback interconnection (Figure 7.1) implies the same for non-zero initial conditions.

We assume that $\phi(0) = 0$ and that ϕ satisfies the somewhat strengthened incremental sector condition

$$\alpha \le \frac{\phi(x_1) - \phi(x_2)}{x_1 - x_2} \le \sup_{x,y \in \mathbb{R}, x \ne y} \frac{\phi(x) - \phi(y)}{x - y} < \beta \ \text{ for all } \ x_1, x_2 \in \mathbb{R}, \ x_1 \ne x_2 \tag{7.4.10}$$

for constants $\alpha \le 0 \le \beta$. This latter constraints makes sure that $\tau\phi$ satisfies the very same properties for all $\tau \in [0,1]$.

Since ϕ is globally Lipschitz the interconnection (7.4.9) is well-posedness. Moreover, (7.4.10) also implies that ϕ is almost everywhere differentiable with $\alpha \le \phi'(x) \le \operatorname{ess\,sup}_{x \in \mathbb{R}} \phi'(x) < \beta$ for almost all $x \in \mathbb{R}$. Recall as well that ϕ satisfies a regular sector condition due to $\phi(0) = 0$.

The following two inequalities are key for the further development.

Lemma 7.26 *For $\sigma \in \mathbb{R}$ and $z \in \mathscr{L}_2$ the signals*

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \phi(z) - \alpha z \\ \beta z - \phi(z) \end{pmatrix} = \Pi_{\alpha,\beta} \begin{pmatrix} \phi(z) \\ z \end{pmatrix} \ \text{ with } \ \Pi_{\alpha,\beta} = \begin{pmatrix} 1 & -\alpha \\ -1 & \beta \end{pmatrix} \tag{7.4.11}$$

satisfy

$$\int_{-\infty}^{\infty} p(t)q(t)dt \ge \int_{-\infty}^{\infty} p(t)q(t-\sigma)dt. \tag{7.4.12}$$

If ϕ is odd (which means $\phi(-x) = -\phi(x)$ for all $x \in \mathbb{R}$) then

$$\int_{-\infty}^{\infty} p(t)q(t)\,dt \ge \left| \int_{-\infty}^{\infty} p(t)q(t-\sigma)dt \right|. \tag{7.4.13}$$

Proof. Since the derivative of $x \to \beta x - \phi(x)$ is positive, this function is strictly increasing and has a strictly increasing inverse which is denoted by ρ. Since $x \to \phi(x) - \alpha x$ is non-decreasing, the same holds for

$$f(\xi) := \phi(\rho(\xi)) - \alpha\rho(\xi).$$

If $\varepsilon > 0$ is chosen with $\phi'(x) \le \beta - \varepsilon$ for almost all $x \in \mathbb{R}$ we can conclude that

$$f'(\xi) = [\phi'(\rho(\xi)) - \alpha]\rho'(\xi) = \frac{\phi'(\rho(\xi)) - \alpha}{\beta - \phi'(\rho(\xi))} \begin{cases} \ge & 0 \\ \le & \frac{\beta - \alpha}{\varepsilon} \end{cases} \ \text{ for all } \ \xi \in \mathbb{R}. \tag{7.4.14}$$

The first inequality implies that the primitive $F(x) = \int_0^x f(\xi)\,d\xi$ of f is convex on $\mathbb{R}$. Therefore $F(\eta) \ge F(\xi) + F'(\xi)(\eta - \xi)$ and hence

$$f(\xi)(\xi - \eta) \ge F(\xi) - F(\eta) \ \text{ for all } \ \xi, \eta \in \mathbb{R}. \tag{7.4.15}$$

The second relation in (7.4.14) and $f(0) = 0$ imply with $\gamma = (\beta - \alpha)/\varepsilon$ that $|f(\xi)| \leq \gamma|\xi|$ for all $\xi \in \mathbb{R}$. Hence $F(0) = 0$ and (7.4.15) for $\eta = 0$ lead to $F(\xi) \leq f(\xi)\xi \leq |f(\xi)\xi| \leq \gamma|\xi|^2$. Due to $F(\xi) \geq 0$ we get

$$|F(\xi)| \leq \gamma|\xi|^2 \quad \text{for all} \ \xi \in \mathbb{R}. \tag{7.4.16}$$

Moreover, if we substitute $\xi = \beta x - \phi(x)$ and $\eta = \beta y - \phi(y)$ in (7.4.15) we can exploit $\rho(\xi) = x$ and $f(\xi) = \phi(x) - \alpha x$ to arrive at

$$(\phi(x) - \alpha x)[(\beta x - \phi(x)) - (\beta y - \phi(y))] \geq F(\beta x - \phi(x)) - F(\beta y - \phi(y)) \quad \text{for all} \ x, y \in \mathbb{R}. \tag{7.4.17}$$

Let us now choose $\sigma \in \mathbb{R}$ and any $z \in \mathscr{L}_2$. Since ϕ satisfies a regular sector condition we infer $\phi(z) \in \mathscr{L}_2$ and hence also $q = \beta z - \phi(z) \in \mathscr{L}_2$ and also $q(\cdot - \sigma) \in \mathscr{L}_2(-\infty, \infty)$. The relation (7.4.16) allows to conclude $F(q) \in \mathscr{L}_1(-\infty, \infty)$ and $F(q(\cdot - \sigma)) \in \mathscr{L}_1(-\infty, \infty)$ with $\mathscr{L}_1(-\infty, \infty)$ denoting the Lebesgue space of real-valued absolutely integrable functions on $\mathbb{R}$. This implies

$$\int_{-\infty}^{\infty} F(q(t)) \, dt - \int_{-\infty}^{\infty} F(q(t - \sigma)) \, dt = 0.$$

If we substitute $x = z(t)$ and $y = z(t - \sigma)$ in (7.4.17) we get

$$p(t)q(t) - p(t)q(t - \sigma) \geq F(q(t)) - F(q(t - \sigma)) \quad \text{for all} \ t \in \mathbb{R}$$

which leads to (7.4.12) after integration.

If ϕ is odd the same holds for $x \to \beta x - \phi(x)$ and hence also for its inverse ρ. Therefore f is odd and F is even. If we substitute $x = z(t)$ and $y = -z(t - \sigma)$ in (7.4.17) we hence

$$p(t)q(t) + p(t)q(t - \sigma) \geq F(q(t)) - F(q(t - \sigma)) \quad \text{for all} \ t \in \mathbb{R}.$$

If integrating and combining with (7.4.12) we arrive at (7.4.13). ∎

How do we arrive at an IQC? By Parseval's theorem and since the time-shift corresponds to multiplication with the exponential function in the frequency domain, inequality (7.4.12) reads as

$$\int_{-\infty}^{\infty} \hat{p}(i\omega)^* \hat{q}(i\omega) \, d\omega \geq \int_{-\infty}^{\infty} \hat{p}(i\omega)^* (e^{-i\omega\sigma}) \hat{q}(i\omega) \, d\omega. \tag{7.4.18}$$

Let us now conically combine these IQCs. This just means to multiply any nonnegative function h, $h(\sigma) \geq 0$ for all $\sigma \in \mathbb{R}$, and integrate. In order to make sure that the involved integrals are finite we actually take $h \in \mathscr{L}_1(-\infty, \infty)$. Let us then recall that

$$\|h\|_1 = \int_{-\infty}^{\infty} |h(\sigma)| \, d\sigma \quad \text{and} \quad \hat{h}(i\omega) = \int_{-\infty}^{\infty} h(\sigma) e^{-i\omega\sigma} \, d\sigma$$

denote the $\mathscr{L}_1$-norm and the Fourier transform of h respectively. Then (7.4.18) implies

$$\int_{-\infty}^{\infty} \hat{p}(i\omega)^* \|h\|_1 \hat{q}(i\omega) \, d\omega \geq \int_{-\infty}^{\infty} \hat{p}(i\omega)^* \hat{h}(i\omega) \hat{q}(i\omega) \, d\omega. \tag{7.4.19}$$

If ϕ is odd then (7.4.13) leads to

$$\int_{-\infty}^{\infty} \hat{p}(i\omega)^* \hat{q}(i\omega)\, d\omega \geq \left| \int_{-\infty}^{\infty} \hat{p}(i\omega)^* (e^{-i\omega\sigma}) \hat{q}(i\omega)\, d\omega \right|.$$

For any real-valued $h \in \mathscr{L}_1(-\infty, \infty)$ we can multiply with $|h|$ to infer

$$\int_{-\infty}^{\infty} \hat{p}(i\omega)^* |h(\sigma)| \hat{q}(i\omega)\, d\omega \geq$$

$$\geq \left| \int_{-\infty}^{\infty} \hat{p}(i\omega)^* (h(\sigma) e^{-i\omega\sigma}) \hat{q}(i\omega)\, d\omega \right| \geq \int_{-\infty}^{\infty} \hat{p}(i\omega)^* (h(\sigma) e^{-i\omega\sigma}) \hat{q}(i\omega)\, d\omega.$$

Integration over $\sigma \in (-\infty, \infty)$ leads again to the very same inequality (7.4.19). If conically combining the IQC (7.4.18) with the static IQCs from (7.15) we finally arrive at the following celebrated family of Zames-Falb multipliers.

Theorem 7.27 *Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ satisfies the incremental sector condition (7.4.10) for $\alpha \leq 0 \leq \beta$. If the nonnegative function $h \in \mathscr{L}_1(-\infty, \infty)$ and $g \in \mathbb{R}$ are related as $\|h\|_1 \leq g$ then*

$$\int_{-\infty}^{\infty} \left(\begin{array}{c} \tau \widehat{\phi(z)}(i\omega) \\ \hat{z}(i\omega) \end{array} \right)^* \Pi_{\alpha,\beta}^T \left(\begin{array}{cc} 0 & g - \hat{h}(i\omega) \\ g - \hat{h}(i\omega)^* & 0 \end{array} \right) \Pi_{\alpha,\beta} \left(\begin{array}{c} \tau \widehat{\phi(z)}(i\omega) \\ \hat{z}(i\omega) \end{array} \right) d\omega \geq 0$$

for all $\tau \in [0,1]$ and all $z \in \mathscr{L}_2$. In the case that ϕ is odd the inequality remains true even if h is not sign-constrained.

Proof. With $w = \tau\phi(z)$ and $\left(\begin{array}{c} \hat{p} \\ \hat{q} \end{array} \right) = \Pi_{\alpha,\beta} \left(\begin{array}{c} \hat{w} \\ \hat{z} \end{array} \right)$ the integrand can be written as

$$\left(\begin{array}{c} \hat{w} \\ \hat{z} \end{array} \right)^* \Pi_{\alpha,\beta}^T \left[\left(\begin{array}{cc} 0 & g - \|h\|_1 \\ g - \|h\|_1 & 0 \end{array} \right) + \left(\begin{array}{cc} 0 & \|h\|_1 - \hat{h} \\ \|h\|_1 - \hat{h}^* & 0 \end{array} \right) \right] \Pi_{\alpha,\beta} \left(\begin{array}{c} \hat{w} \\ \hat{z} \end{array} \right) =$$

$$= (g - \|h\|_1) \left(\begin{array}{c} \hat{w} \\ \hat{z} \end{array} \right)^* \left(\begin{array}{cc} -1 & \frac{\alpha+\beta}{2} \\ \frac{\alpha+\beta}{2} & -\alpha\beta \end{array} \right) \left(\begin{array}{c} \hat{w} \\ \hat{z} \end{array} \right) + \hat{p}^*(\|h\|_1 - \hat{h})\hat{q} + \hat{q}^*(\|h\|_1 - \hat{h}^*)\hat{p}.$$

After integration over frequency and using Parseval's theorem, we get

$$2\pi(g - \|h\|_1) \int_0^{\infty} \left(\begin{array}{c} w(t) \\ z(t) \end{array} \right)^T \left(\begin{array}{cc} -1 & \frac{\alpha+\beta}{2} \\ \frac{\alpha+\beta}{2} & -\alpha\beta \end{array} \right) \left(\begin{array}{c} w(t) \\ z(t) \end{array} \right) dt + 2 \int_{-\infty}^{\infty} \hat{p}(i\omega)^* (\|h\|_1 - \hat{h}(i\omega)) \hat{q}(i\omega) d\omega$$

where we exploited the fact that the latter integral is real. Non-negativity is obtained from Example 7.15 and (7.4.19). ∎

Stability is guaranteed if we can find g and h with the properties as in Theorem 7.27 which satisfy, for some $\varepsilon > 0$, (7.4.1) or equivalently the FDI

$$\left(\begin{array}{c} 1 \\ T(i\omega) \end{array} \right)^* \Pi_{\alpha,\beta}^T \left(\begin{array}{cc} 0 & g - \hat{h}(i\omega) \\ g - \hat{h}(i\omega)^* & 0 \end{array} \right) \Pi_{\alpha,\beta} \left(\begin{array}{c} 1 \\ T(i\omega) \end{array} \right) \preccurlyeq -\varepsilon I \quad \text{for all} \quad \omega \in \mathbb{R}.$$

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Since both $T(i\omega)$ and $\hat{h}(i\omega)$ (by the Riemann-Lebesgue Lemma) are continuous in $\omega \in [0,\infty]$ this condition can be equivalently replaced by

$$\left(\begin{array}{c} 1 \\ T(i\omega) \end{array} \right)^* \Pi_{\alpha,\beta}^T \left(\begin{array}{cc} 0 & g - \hat{h}(i\omega) \\ g - \hat{h}(i\omega)^* & 0 \end{array} \right) \Pi_{\alpha,\beta} \left(\begin{array}{c} 1 \\ T(i\omega) \end{array} \right) \prec 0 \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}. \tag{7.4.20}$$

Remark 7.28 Suppose that $g \in \mathbb{R}$ and $h \in \mathscr{L}_1(-\infty,\infty)$ satisfy (7.4.20) and $\|h\|_1 \le g$. For reasons of continuity we can replace g by some slightly larger value without violating (7.4.20). Without loss of generality the stability test can hence be based on $\|h\|_1 < g$.

Now suppose that, in addition, $h(t) \ge 0$ for all $t \in \mathbb{R}$. Let us argue why this can be replaced with the strict inequality $h(t) > 0$ for $t \in \mathbb{R}$. For this purpose recall that

$$\|\hat{h}_1 - \hat{h}_2\|_\infty = \max_{\omega \in \mathbb{R} \cup \{\infty\}} \left| \hat{h}_1(i\omega) - \hat{h}_2(i\omega) \right| \le \|h_1 - h_2\|_1 \ \text{ for } h_1, h_2 \in \mathscr{L}_1(-\infty,\infty).$$

For example with $h_\delta(t) = \delta e^{-|t|}$, $\delta > 0$, we have $h(t) + h_\delta(t) > 0$ for all $t \in \mathbb{R}$; moreover $\|\hat{h} - (\hat{h} + \hat{h}_\delta)\|_\infty \le \|h_\delta\|_1 \to 0$ for $\delta \to 0$, which reveals that (7.4.20) persists to hold for $h + h_\delta$ if only $\delta > 0$ is sufficiently small.

In summary, for computations we can impose without loss of generality the constraints

$$h \in \mathscr{L}_1(-\infty,\infty), \ g \in \mathbb{R}, \ \|h\|_1 < g \ (\text{and} \ h(t) > 0 \ \text{for all} \ t \in \mathbb{R}).$$

The choice $h = 0$ leads to the previously derived tests based on static IQCs, just because

$$\Pi_{\alpha,\beta}^T \left(\begin{array}{cc} 0 & g \\ g & 0 \end{array} \right) \Pi_{\alpha,\beta} = \left(\begin{array}{cc} -2g & g(\alpha+\beta) \\ g(\alpha+\beta) & -2g\alpha\beta \end{array} \right) \ \text{ and } \ g > 0.$$

Let us now generalize the search to rational functions $\hat{h}$ which must be strictly proper (since $\hat{h}$ vanishes at infinity for $h \in \mathscr{L}_1(-\infty,\infty)$ due to the Riemann-Lebesgue Lemma). Let ψ be any vector of strictly proper rational functions without pole on the imaginary axis. We then parameterize $\hat{h}$ as $c\psi$ with real row vectors c. If we define

$$H(c,g) = \left(\begin{array}{ccc} 0 & g & -c \\ g & 0 & 0 \\ -c^T & 0 & 0 \end{array} \right) \ \text{ and } \ \Psi = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & \psi \end{array} \right) \Pi_{\alpha,\beta} \tag{7.4.21}$$

we clearly have

$$\Pi_{\alpha,\beta}^T \left(\begin{array}{cc} 0 & g - \hat{h}(i\omega) \\ g - \hat{h}(i\omega)^* & 0 \end{array} \right) \Pi_{\alpha,\beta} = \Psi^* H(c,g) \Psi.$$

Therefore the set of Zames-Falb multipliers is actually described by a fixed (typically tall) transfer matrix Ψ and a family of structured real symmetric (indefinite) matrices $H(c,g)$ that depend affinely on (c,g). With a state-space realization of $\Psi\,\mathrm{col}(1,T)$, the FDI (7.4.20) can be translated into an LMI constraint on (c,g). If $\check{\psi}$ denotes the inverse Fourier transform of ψ, it then remains to turn $g > \|c\check{\psi}\|_1$ (and $c\check{\psi}(t) > 0$ for all $t \in \mathbb{R}$ if the nonlinearity is not odd) into an LMI constraint for actual computations. Note that [**?**, Table 1] provides a list of suitable basis functions that can be employed in this context.

a	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8
b_{cir}	2.19	1.6	1.24	1	0.83	0.705	0.611	0.537
b_{zfc}	3.12	2.21	1.67	1.32	1.08	0.905	0.776	0.677
$b_{\text{zf}} = b_{\text{par}}$	4.55	2.91	2.1	1.62	1.31	1.09	0.925	0.803

Table 7.2: Computed stability margins in Example 7.29.

Example 7.29 We continue Example 7.20. Let us parameterize the multipliers either with

$$\psi(s) = \frac{1}{s+1} \ \text{ or with } \ \psi(s) = \frac{1}{s-1}$$

that both satisfy $\|\check{\psi}\|_1 = 1$. Since the saturation is odd-monotonous, we need to impose the conditions $g - \|c\check{\psi}\|_1 = g - |c| > 0$ on the real constants g and c which is LMIable. Observe that the former multiplier is causal while the latter is non-causal. In classical papers some of the main difficulties in proofs arise from non-causality since the reduction to an application of the passivity theorem requires somewhat deeper results from factorization theory. The IQC approach avoids all these difficulties and applies to both multiplier parameterizations directly. The computed stability margins b_{zfc} and b_{zf} for the causal and non-causal versions are given in Table 7.29. We observe that causal dynamics improve over the circle criterion, while non-causal dynamics hit the parametric stability margin b_{par}. This implies that b_{zf} is indeed the largest possible value of b for which the interconnection of Example 7.20 can be stable, no matter which other technique might be applied.

We conclude this section by providing a general parameterization of the set of Zames-Falb multipliers that is amenable to LMI computations. For this purpose let us collect some key classical facts from approximation theory. Fix $a > 0$. It is then well-known that

$$e^{-at} p(t) \ \text{ with } p \text{ being any polynomial}$$

can approximate functions in $\mathscr{L}_2[0, \infty)$ and $\mathscr{L}_2$ arbitrarily closely [**?**]. It is relevant to observe that non-negative functions $\mathscr{L}_2[0, \infty)$ can be approximated in this fashion with positive polynomials p. We owe the proof of this fact to Jonathan Partington [**?**].

Lemma 7.30 *Let* $h \in \mathscr{L}_2[0, \infty)$ *satisfy* $h(t) \geq 0$ *for* $t \geq 0$. *For all* $\varepsilon > 0$ *there exists a polynomial* p *such that, with* $q(t) = e^{-at} p(t)$, *we have*

$$\|h - q\|_1 < \varepsilon \ \text{ and } \ p(t) > 0 \ \text{ for all } \ t \geq 0.$$

Proof. If $h(t) \geq 0$ we can define the function $f(t) = \sqrt{h(t)}$ which is in $\mathscr{L}_2$. Hence there exists some polynomial $\tilde{p}$ such that the $\mathscr{L}_2$ norm of $f(t) - \tilde{p}(t)e^{-at/2}$ is smaller than $\sqrt{\varepsilon + \|f\|^2} - \|f\|$. Then

the $\mathscr{L}_2$-norm of $f(t) + \tilde{p}(t)e^{-at/2}$ is not larger than $\sqrt{\varepsilon + \|f\|^2} + \|f\|$. Therefore

$$\int_0^\infty |h(t) - \tilde{p}(t)^2 e^{-at}|\,dt = \int_0^\infty |f(t) - \tilde{p}(t)e^{-at/2}|\,|f(t) + \tilde{p}(t)e^{-at/2}|\,dt \leq$$

$$\leq \sqrt{\int_0^\infty |f(t) - \tilde{p}(t)e^{-at/2}|^2\,dt}\,\sqrt{\int_0^\infty |f(t) + \tilde{p}(t)e^{-at/2}|\,dt} <$$

$$< (\sqrt{\varepsilon + \|f\|^2} - \|f\|)(\sqrt{\varepsilon + \|f\|^2} + \|f\|) = \varepsilon.$$

For a sufficiently small $\tilde{\varepsilon}$ this persists to hold for $p := \tilde{p}^2 + \tilde{\varepsilon}$ which is a polynomial that is positive on $[0,\infty)$ as required.   ∎

We will as well make beneficial use of a specialization of a less commonly known result on one-sided $\mathscr{L}_2$-approximations that is due to Géza Freud [**?**, **?**].

Lemma 7.31 *Let p be any polynomial and define $q(t) = e^{-at}|p(t)|$. For any $\varepsilon > 0$ there exists some $\tilde{p}$ such that, with $\tilde{q}(t) = e^{-at}\tilde{p}(t)$, we have*

$$\|q - \tilde{q}\|_1 < \varepsilon \ \text{ and } \ |p(t)| < \tilde{p}(t) \ \text{ for all } \ t \geq 0.$$

Let us now describe a generic parameterization of Zames-Falb multipliers. For this purpose we denote by J_N the standard upper triangular Jordan block of dimension $N \times N$ with eigenvalue zero and define the Hurwitz matrix $A_N = -aI_N + J_N$. If B_N is the column vector of length N whose only non-zero component is 1 in the last position, we use the abbreviations

$$P_N(t) = e^{J_N t} B_N = \text{col}\left(\frac{t^{N-1}}{(N-1)!}, \dots, \frac{t^2}{2!}, t, 1\right) \ \text{ and } \ Q_N(t) := e^{A_N t} B_N = e^{-at} P_N(t).$$

Since $-A_N^{-1}B_N$ equals the all-ones vector E_N let us record for later purpose that

$$\int_0^\infty Q_N(t)\,dt = E_N. \tag{7.4.22}$$

Clearly the components of P_N form a basis of the space of all polynomials with degree at most $N-1$. Hence for $N \in \mathbb{N}$ and a free row vector $C \in \mathbb{R}^{1 \times N}$, the functions $CQ_N(t) = e^{-at}CP_N(t)$ are those exponentially weighted polynomials that were discussed in our introductory remarks. This motivates the parameterization of $h \in \mathscr{L}_2[0,\infty)$ as

$$Ce^{A_N t} B_N = CQ_N(t) \ \text{ for } \ t \geq 0. \tag{7.4.23}$$

Positivity is guaranteed if CP_N is positive on $[0,\infty)$. Due to (7.4.22), the $\mathscr{L}_2[0,\infty)$-norm of (7.4.23) is then smaller than g iff $C^T E_N < g$. Hence (7.4.23) serves as a Zames-Falb multiplier for general non-linearities if the coefficient vector C satisfies the additional constraints

$$CP_N(t) > 0 \ \text{ for all } \ t \geq 0 \ \text{ and } \ C^T E_N < g. \tag{7.4.24}$$

These constraints can be turned into LMIs. Moreover the suggested parametrization is tight in the following sense.

Lemma 7.32 *Suppose that there exist $g > 0$ and $h \in \mathscr{L}_2[0,\infty)$ with $h(t) > 0$ for $t \geq 0$ and $\|h\|_1 < g$ that satisfy the FDI (7.4.20). Then there exist $N \in \mathbb{N}$ and $C \in \mathbb{R}^{1 \times N}$ with (7.4.24) such that the FDI (7.4.20) is valid for $h = CQ_N$.*

Proof. Let h be as described. Then there exists some $\delta > 0$ such that the FDI (7.4.20) persists to hold for all other functions in $\mathscr{L}_2[0,\infty)$ whose Fourier transform deviates by at most δ from $\tilde{h}$ in the peak-norm over frequency. We can choose δ sufficiently small to also guarantee $\|h\|_1 < g - \delta$. By Lemma 7.30 there exist $N \in \mathbb{N}$ and $C \in \mathbb{R}^{1 \times N}$ with $\|h - CQ_N\|_1 < \delta$ and $CQ_N(t) > 0$ for all $t \geq 0$. Hence $\|CQ_N\|_1 \leq \|CQ_N - h\|_1 + \|h\|_1 < g$ which implies (7.4.24). Moreover we infer from $\|\tilde{h} - \widehat{CQ_N}\|_\infty < \delta$ that $\widehat{CQ_N}$ satisfies (7.4.20). ∎

If h is not constrained in sign one needs to translate

$$\int_0^\infty |CQ_N(t)|\, dt < g$$

into an LMI constraint. This is assured if there exists some $D \in \mathbb{R}^{1 \times N}$ with

$$|CP_N(t)| < DP_N(t) \ \text{ for } \ t \geq 0 \ \text{ and } \ D^T E_N < g. \tag{7.4.25}$$

Indeed the first relation implies $|CQ_N(t)| < DQ_N(t)$ for all $t \geq 0$; since $DQ_N(t)$ is positive for $t \geq 0$ we can use (7.4.22) to infer $\|DQ_N\|_1 = D^T E_N$ and hence, by the second relation, $\|DQ_N\|_1 < g$. It is not difficult to see that, again, this parameterization is tight.

Lemma 7.33 *Suppose that there exist $g > 0$ and $h \in \mathscr{L}_2[0,\infty)$ with $\|h\|_1 < g$ that satisfy the FDI (7.4.20). Then there exist $N \in \mathbb{N}$ and $C, D \in \mathbb{R}^{1 \times N}$ with (7.4.24) such that the FDI (7.4.20) is valid for $h = CQ_N$.*

Proof. Let h be as described and choose δ as in the proof of Lemma 7.4.18. Then there exist $M \in \mathbb{N}$ and $C \in \mathbb{R}^{1 \times M}$ with $\|h - CQ_M\|_1 < \delta/2$. Hence (7.4.20) persists to hold for $\hat{h}$ being replaced by $\widehat{CQ_M}$ and $\|CQ_M\|_1 \leq \|CQ_M - h\|_1 + \|h\|_1 < g - \delta/2$. By Lemma 7.31 we can then find some $N \geq M$ and $D_M \in \mathbb{R}^{1 \times M}$ with $\||CQ_M| - DQ_N\|_1 < \delta/2$ and $CP_M(t) < DP_N(t)$ for all $t \geq 0$. We then infer $\|DQ_N\|_1 \leq \|DQ_N - |CQ_M|\|_1 + \|CQ_M\|_1 < g$ which implies (7.4.25) due to (7.4.22). The proof is finished by observing that CQ_M can be written as CQ_N by padding C with zeros. ∎

The parameterization of $h \in \mathscr{L}_1(-\infty,\infty)$ is obtained by concatenating functions in $\mathscr{L}_1(-\infty,0)$ with functions in $\mathscr{L}_1[-\infty,0)$ as

$$h(t) = \begin{cases} C_- e^{-A_M t} B_M & \text{for} \quad t < 0 \\ C_+ e^{A_N t} B_N & \text{for} \quad t \geq 0 \end{cases}$$

for $M, N \in \mathbb{N}$ and $C_- \in \mathbb{R}^{1 \times M}$, $C_+ \in \mathbb{R}^{1 \times N}$. Let us introduce the abbreviations

$$c = \begin{pmatrix} C_- & C_+ \end{pmatrix}, \ \psi(s) = \begin{pmatrix} -(sI + A_M)^{-1} B_M \\ (sI - A_N)^{-1} B_N \end{pmatrix}, \ p(t) = \begin{pmatrix} P_M(-t) \\ P_N(t) \end{pmatrix}, \ e = \begin{pmatrix} E_M \\ E_N \end{pmatrix}$$
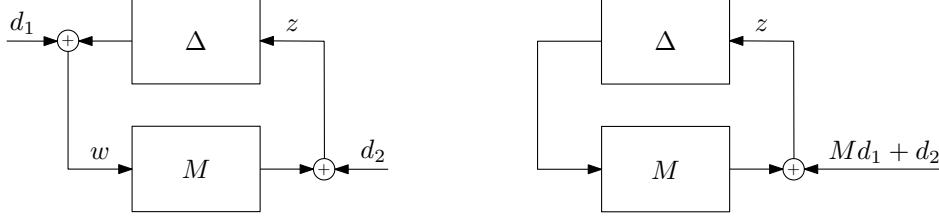
Figure 7.5: Specific Feedback Interconnection III

and again (7.4.21). Then the Fourier-transform of h is

$$\hat{h}(i\omega) = -C_-(i\omega I + A_M)^{-1}B_M + C_+(i\omega I - A_N)^{-1}B_N = c\psi(i\omega).$$

Therefore, (7.4.20) reads as

$$\left(\begin{array}{c} 1 \\ T(i\omega) \end{array} \right)^* \Psi(i\omega)^* H(c,g)\Psi(i\omega) \left(\begin{array}{c} 1 \\ T(i\omega) \end{array} \right) \prec 0 \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}$$

which can be translated into an LMI constraint on (c,g) with the KYP-Lemma. Moreover, for general non-linearities we can assure $\|h\|_1 < g$ and $h(t) > 0$ for all $t \in \mathbb{R}$ by

$$ce < g \ \text{ and } \ cp(t) > 0 \ \text{ for } \ t \geq 0.$$

Finally for odd-monotone nonlinearities the sole constraint $\|h\|_1 < g$ is guaranteed if there exists some $d \in \mathbb{R}^{1 \times (M+N)}$ with $|cp(t)| < dp(t)$ or

$$de < g \ \text{ and } \ \left(\begin{array}{cc} dp(t) & cp(t) \\ cp(t) & dp(t) \end{array} \right) \succ 0 \ \text{ for } \ t \geq 0.$$

All these constraints are LMIable.

7.4.3   General version of main QC stability result

The purpose of this section is to generalize Theorem 7.24 in various respects. We consider the configuration in Figure 7.5 as described by

$$w = \Delta(z) + d_1 \ \text{ and } \ z = Mw + d_2,$$

which reduces to the earlier interconnection

$$z = Mw + (Md_1 + d_2) \ \text{ and } \ w = \Delta(z)$$

with external input $d = Md_1 + d_2$.

Figure 7.6: Specific Feedback Interconnection IV

In precise terms we assume that $M : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^l$ and $\Delta : \mathscr{L}_{2e}^l \to \mathscr{L}_{2e}^k$ are causal and bounded while M is linear. The external disturbance d is supposed to be confined to the set $M\mathscr{L}_{2e}^k + N\mathscr{V}_e$ with some bounded linear filter N and some subset $\mathscr{V}_e \subset \mathscr{L}_{2e}^m$. Consistently with the convention up to now we denote by $\mathscr{V} := \mathscr{V}_e \cap \mathscr{L}_2^m$ the finite energy signals in $\mathscr{V}_e$. Note that it is not excluded to have $\mathscr{V} \subset \mathscr{L}_2^m$ which implies $\mathscr{V}_e = \mathscr{V}$. The feedback system under consideration as depicted in Figure 7.6 is described by

$$z - M\Delta(z) = Mu + Nv \quad \text{with } (u, v) \in \mathscr{L}_{2e}^k \times \mathscr{V}_e. \tag{7.4.26}$$

Let us introduce the notions of well-posedness and stability that is used in this context.

Definition 7.34 The interconnection (7.4.26) is *well-posed* if for each $(u, v) \in \mathscr{L}_{2e}^k \times \mathscr{V}_e$ there exists a unique $z \in \mathscr{L}_{2e}^l$ satisfying $z - M\Delta(z) = Mu + Nv$ and such that the correspondingly defined **response map** $(u, v) \to z = S(u, v)$ is causal in the first argument:

$$S(u, v)_T = S(u_T, v)_T \quad \text{for all } T > 0, \ (u, v) \in \mathscr{L}_{2e}^k \times \mathscr{V}_e.$$

The feedback system (7.4.26) is **stable** if, in addition, $S : \mathscr{L}_2^k \times \mathscr{V} \to \mathscr{L}_{2e}^l$ is bounded.

Well-posedness is clearly a consequence of $I - M\Delta$ having a causal inverse since then $S(u, v) = (I - M\Delta)^{-1}(Mu + Nv)$. Stability is in turn implied by boundedness if $(I - M\Delta)^{-1}$.

For the map Σ we now only require the existence of $\sigma_{ij} \in \mathbb{R}$ with

$$\Sigma \begin{pmatrix} w \\ Mw + Mu + Nv \end{pmatrix} - \Sigma \begin{pmatrix} w \\ Mw \end{pmatrix} \leq$$

$$\leq \begin{pmatrix} \|w\| \\ \|u\| \\ \|v\| \end{pmatrix}^T \begin{pmatrix} 0 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \begin{pmatrix} \|w\| \\ \|u\| \\ \|v\| \end{pmatrix} \quad \text{for all } w, u \in \mathscr{L}_2^k, \ v \in \mathscr{V}. \tag{7.4.27}$$

This is indeed true if Σ is a bounded quadratic form as easily seen along the lines of the first step in the proof of Theorem 7.24.

Theorem 7.35 *Suppose that*

- *the feedback system (7.4.26) is well-posed for all $\tau\Delta$ with $\tau \in [0, 1]$;*

- *there exists $\varepsilon > 0$ such that*

$$\Sigma \begin{pmatrix} w \\ Mw \end{pmatrix} \leq -\varepsilon\|w\|^2 \ \text{for all } w \in \mathscr{L}_2^k; \tag{7.4.28}$$

- *there exists $\delta_0 : \mathcal{V} \to \mathbb{R}$ with*

$$\Sigma \begin{pmatrix} \tau\Delta(z) \\ z \end{pmatrix} \geq -\delta_0(v) \text{ for all } \tau \in [0,1], \ z = Mu + Nv, \ (u,v) \in \mathcal{L}_2^k \times \mathcal{V}. \tag{7.4.29}$$

Then there exists some $\gamma > 0$ (only depending on M and Σ) such that

$$\|S(u,v)\|^2 \leq \gamma^2 \left(\|u\|^2 + \|v\|^2 \right) + \gamma\delta_0(v) \text{ for all } u \in \mathcal{L}_2^k, \ v \in \mathcal{V}. \tag{7.4.30}$$

Proof. For $\tau \in [0,1]$ let $z = S_\tau(u,v)$ denote the response of (7.4.26) if Δ is replaced by $\tau\Delta$. Moreover abbreviate $\mathcal{D} := \mathcal{L}_2^k \times \mathcal{V}$ and $\mathcal{D}_e := \mathcal{L}_{2e}^k \times \mathcal{V}_e$.

Step 1. Only for proving the following key fact we make use of the properties of Σ and the two hypothesized quadratic constraints: There exists a (τ-independent) $\gamma > 0$ such that

$$\tau \in [0,1] \text{ and } S_\tau(\mathcal{D}) \subset \mathcal{L}_2^l \text{ imply}$$
$$\|S_\tau(u,v)\|^2 \leq \gamma^2 \left(\|u\|^2 + \|v\|^2 \right) + \gamma\delta_0(v) \text{ for all } (u,v) \in \mathcal{D}. \tag{7.4.31}$$

Observe for all $\gamma > 0$ and all $w \in \mathcal{L}_2^k$, $(u,v) \in \mathcal{D}$ that

$$\frac{1}{\gamma}\|Mw + Mu + Nv\|^2 - \gamma \left(\|u\|^2 + \|v\|^2 \right) \leq$$

$$\leq \frac{1}{\gamma} \left(\|M\|\|w\| + \|M\|\|u\| + \|N\|\|v\| \right)^2 - \gamma \left(\|u\|^2 + \|v\|^2 \right) =$$

$$= \begin{pmatrix} \|w\| \\ \|u\| \\ \|v\| \end{pmatrix}^T \begin{pmatrix} m_{11}/\gamma & m_{12}/\gamma & m_{13}/\gamma \\ m_{12}/\gamma & m_{22}/\gamma - \gamma & m_{23}/\gamma \\ m_{13}/\gamma & m_{23}/\gamma & m_{33}/\gamma - \gamma \end{pmatrix} \begin{pmatrix} \|w\| \\ \|u\| \\ \|v\| \end{pmatrix}, \tag{7.4.32}$$

where m_{ij} only depend on $\|M\|$ and $\|N\|$. If we add (7.4.27), (7.4.28) and (7.4.32) one shows exactly as in Step 1 of the proof of Theorem 7.16 that there exists some $\gamma > 0$ with

$$\Sigma \begin{pmatrix} w \\ Mw + Mu + Nv \end{pmatrix} + \frac{1}{\gamma}\|Mw + Mu + Nv\|^2 - \gamma \left(\|u\|^2 + \|v\|^2 \right) \leq 0 \tag{7.4.33}$$

for all $w \in \mathcal{L}_2^k$, $(u,v) \in \mathcal{D}$.

Now fix any $(u,v) \in \mathcal{D}$. Due to the hypothesis in (7.4.31), we infer that $z := S_\tau(u,v)$ satisfies $z \in \mathcal{L}_2^l$ and thus also $w := \tau\Delta(z) \in \mathcal{L}_2^k$. Since we also have $z = Mw + Mu + Nv$ by (7.4.26), we can exploit (7.4.33) to get

$$\Sigma \begin{pmatrix} \tau\Delta(M(w+u) + Nv) \\ M(w+u) + Nv \end{pmatrix} + \frac{1}{\gamma}\|S_\tau(u,v)\|^2 - \gamma \left(\|u\|^2 + \|v\|^2 \right) \leq 0. \tag{7.4.34}$$

Since $w + u \in \mathcal{L}_2^k$ and $v \in \mathcal{V}$ it remains to use (7.4.29) in order to obtain from (7.4.34) the inequality in (7.4.31).

Step 2. There exists $\tilde{\delta} > 0$, $\tilde{\delta}_0 \geq 0$ such that $\|\Delta(z)_T\| \leq \tilde{\delta}\|z_T\| + \tilde{\delta}_0$ for all $T > 0$, $z \in \mathscr{L}_{2e}^l$. With $\gamma > 0$ from Step 1 fix any $\rho_0 > 0$ with $\gamma\rho_0\tilde{\delta} < 1$. In this step we show that

$$\tau \in [0,1], \ \tau + \rho \in [0,1], \ |\rho| \leq \rho_0, \ S_\tau(\mathscr{D}) \subset \mathscr{L}_2^l \ \text{imply} \ S_{\tau+\rho}(\mathscr{D}) \subset \mathscr{L}_2^l. \tag{7.4.35}$$

Fix ρ and τ as in (7.4.35) and any $(u,v) \in \mathscr{D}$. We have to show that $z = S_{\tau+\rho}(u,v) \in \mathscr{L}_2^l$. Observe that $z - \tau M\Delta(z) - \rho M\Delta(z) = Mu + Nv$ or $z - \tau M\Delta(z) = M(\rho\Delta(z) + u) + Nv$ or

$$z = S_\tau(\rho\Delta(z) + u, v).$$

Again the key idea is to just employ a small-gain argument based on $1 - \gamma\rho_0\tilde{\delta} > 0$ as follows. The hypothesis in (7.4.35) allows to exploit (7.4.31); if we recall that S_τ is causal in the first argument, we infer with $\gamma_0 = \sqrt{\gamma\delta_0(v)}$ for $T > 0$ that

$$\|z_T\| = \|S_\tau(\rho\Delta(z) + u, v)_T\| = \|S_\tau(\rho\Delta(z)_T + u_T, v)_T\| \leq$$
$$\leq \|S_\tau(\rho\Delta(z)_T + u_T, v)\| \leq \gamma\|\rho\Delta(z)_T + u_T\| + \gamma\|v\| + \gamma_0 \leq$$
$$\leq \gamma|\rho|(\tilde{\delta}\|z_T\| + \tilde{\delta}_0) + \gamma\|u_T\| + \gamma\|v\| + \gamma_0 \leq$$
$$\leq (\gamma\rho_0\tilde{\delta})\|z_T\| + \gamma\|u\| + \gamma\|v\| + \gamma\rho_0\tilde{\delta}_0 + \gamma_0.$$

Hence $(1 - \gamma\rho_0\tilde{\delta})\|z_T\| \leq \gamma\|u\| + \gamma\|v\| + \gamma\rho_0\tilde{\delta}_0 + \gamma_0$ for all $T > 0$ which implies $z \in \mathscr{L}_2^l$.

Step 3. Clearly $S_0(\mathscr{D}) \subset \mathscr{L}_2^l$. Since ρ_0 in Step 2 does not depend on τ, we can inductively apply (7.4.35) in order to infer $S_\tau(\mathscr{D}) \subset \mathscr{L}_2^l$ for $\tau \in [0, \nu\rho_0] \cap [0,1]$ and all $\nu = 1, 2, \ldots$, and thus in particular also for $\tau = 1$. Hence (7.4.31) implies (7.4.30). ∎

Let us highlight the important specialization to $\mathscr{V} = \mathscr{L}_{2e}^k$ if N and S are causal. If $\delta_0(v) = \delta_0$ is constant, (7.4.30) just implies that the whole response map S is bounded. With a further specialization to $N = I$ a little reflection reveals that one recovers Theorem 7.24.

Remark 7.36

- Hence Theorem 7.24 is just a corollary to Theorem 7.35, with the proof of the former being very similar and not considerably harder than that of the latter. We opted to separate both the formulation of the theorems and their proofs for didactical reasons only. It should be stressed that Theorem 7.24 is the precise analogue of the main result of [**?**] as formulated for IQCs, while Theorem 7.35 extends [**?**], which is devoted to an IQC theorem with Popov multipliers, to an abstract setting.

- The proof of the theorem proceeds via a homotopy argument in order to show that $S_\tau(\mathscr{L}_2^k, \mathscr{V})$ does not leave the space $\mathscr{L}_2^l$ if τ moves from 0 to 1. Generalizations in which one replaces the line-segment $\{\tau\Delta : \ \tau \in [0,1]\}$ between 0 and Δ with any continuous curve in the set of bounded casual operators are easy to formulate and to prove. In a similar vein, we stress that advanced readers should be in the position to generalize the arguments in the proof to more sophisticated stability question that are not directly covered by Theorem 7.35.

7.4.4   Popov criteria

As a matter of motivation for the Popov criterion let us consider the nonlinear system

$$\dot{x} = Ax + B\phi(Cx), \quad x(0) = x_0 \tag{7.4.36}$$

with A being Hurwitz and with $\phi : \mathbb{R} \to \mathbb{R}$ being continuously differentiable and satisfying a sector condition with constants $\alpha = 0$ and $\beta > 0$. Our goal is improve on the circle criterion based on static IQCs by the inclusion of suitable dynamics in the multiplier.

Classically, the key idea is based on the following observation. For any $\lambda \geq 0$ the series interconnection of the filter $1/(\lambda s + 1)$ with the non-linearity ϕ as described by

$$\xi(0) = \xi_0, \quad \lambda \dot{\xi} = -\xi + u, \quad y = \phi(\xi) \tag{7.4.37}$$

is passive. In order to show this property let us introduce

$$I_\phi(x) = \int_0^x \phi(s)\, ds \;\; \text{with satisfies} \;\; \frac{d}{dx} I_\phi(x) = \phi(x) \;\; \text{and} \;\; I_\phi(x) \geq 0 \;\; \text{for all} \;\; x \in \mathbb{R},$$

the latter being a consequence of $\phi(s) \geq 0$ for $s \geq 0$ and $\phi(s) \leq 0$ for $s \leq 0$ due to the sector condition. For any trajectory of (7.4.37) we hence have $\xi(t)\phi(\xi(t)) \geq 0$ which allows to infer for all $T > 0$:

$$\int_0^T u(t)y(t)\mathrm{d}t = \int_0^T [\lambda \dot{\xi}(t) + \xi(t)]\phi(\xi(t))\, dt \geq \lambda \int_0^T \dot{\xi}(t)\phi(\xi(t))\, dt =$$

$$= \lambda \int_0^T \frac{d}{dt} I_\phi(\xi(t))\, dt = \lambda I_\phi(\xi(T)) - \lambda I_\phi(\xi(0)) \geq -\lambda I_\phi(\xi_0).$$

We now observe, for any trajectory of (7.4.36) and with $z := Cx$ and $w := \phi(z)$, that $\zeta := \lambda \frac{d}{dt} z + z$ satisfies $\zeta = (\lambda CA + C)x + \lambda CBw$. Moreover $z(0) = Cx_0$. Therefore the x-trajectories of (7.4.36) are same as those of

$$\begin{cases} \dot{x} = Ax + Bw, \qquad\qquad x(0) = x_0 \\ \zeta = (\lambda CA + C)x + \lambda CBw \end{cases} \quad \text{interconnected with} \quad \begin{cases} \lambda \dot{\xi} = -\xi + \zeta, \qquad \xi(0) = Cx_0 \\ w = \phi(\xi) \end{cases}$$

$$\tag{7.4.38}$$

By the passivity theorem (and in view of Section 7.3.3), interconnection stability is guaranteed if the former LTI system is strictly anti-passive for some $\lambda \geq 0$. With $T(s) = C(sI - A)^{-1}B$ observe that its transfer function is $(1 + \lambda s)T(s)$. Strict anti-passivity hence translates into the FDI

$$\mathrm{Re}[(1 + \lambda i\omega)T(i\omega)] < 0 \;\; \text{for all} \;\; \omega \in \mathbb{R} \cup \{\infty\}$$

which is in turn equivalent to feasibility of the LMI

$$\begin{pmatrix} A^T X + XA & XB \\ B^T X & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ \lambda CA + C & \lambda CB \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda CA + C & \lambda CB \end{pmatrix} \prec 0.$$

We have derived some classical version of the Popov criterion for (7.4.36) which is based on hard IQCs that require the parameter λ to be non-negative.

Let us now discuss how soft IQCs allow to overcome this limitation with ease. In fact we derive a general version of the Popov criterion for

$$
\begin{cases}
\dot{x} = Ax + Bw, & x(0) = x_0 \\
w = \Delta(z) + u \\
z = Cx
\end{cases}
\tag{7.4.39}
$$

with A being Hurwitz and with the sector-bounded diagonal nonlinearity Δ in Example 7.15. Since (7.4.39) is a specialization of (7.3.10), we define (7.3.12) and investigate the stability of (7.3.13) with the set-up of Section 7.4.3 and the choices

$$
N(v) = Cv \ \text{ and } \ \mathscr{V} = \{e^{A\cdot}x_0\}.
$$

Again in view of the discussion in Section 7.3.3 and by Example 7.10, the required well-posedness property is assured for all $\tau\Delta$ with $\tau \in [0,1]$ and for all $x_0 \in \mathbb{R}^n$. Moreover the input-output and input-state stability properties of Section 7.3.3 hold if one can show the existence of constants γ and γ_0 such that all trajectories of (7.4.39) satisfy

$$
\|z\|^2 \le \gamma^2 \|u\|^2 + \gamma_0^2 \|x_0\|^2.
\tag{7.4.40}
$$

Let us note that $z = Mu + Nv$ for $u \in \mathscr{L}_2^k$ and $v \in \mathscr{V}$ satisfies

$$
\begin{pmatrix} \dot{\xi} \\ z \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A & B \\ C & 0 \\ CA & CB \end{pmatrix} \begin{pmatrix} \xi \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ N \\ NA \end{pmatrix} v, \ \ \xi(0) = 0.
\tag{7.4.41}
$$

Hence $z \in \mathscr{L}_2^l$ is differentiable and $\dot{z} \in \mathscr{L}_2^l$. For any $\lambda \in \mathbb{R}^k$ and as motivated by our preliminary considerations we can hence define the map

$$
\Sigma_\lambda^{\mathrm{Popov}} \begin{pmatrix} w \\ z \end{pmatrix} = \int_0^\infty w(t)^T \operatorname{diag}(\lambda)\dot{z}(t)\,dt = \int_{-\infty}^\infty \widehat{w}(i\omega)^* \operatorname{diag}(\lambda)(i\omega)\widehat{z}(i\omega)\,d\omega =
$$

$$
= \int_{-\infty}^\infty \begin{pmatrix} \widehat{w}(i\omega) \\ \widehat{z}(i\omega) \end{pmatrix}^* \Pi_\lambda^{\mathrm{Popov}}(i\omega) \begin{pmatrix} \widehat{w}(i\omega) \\ \widehat{z}(i\omega) \end{pmatrix} d\omega \ \text{ for } w \in \mathscr{L}_2^l, \ z \in M\mathscr{L}_2^k + N\mathscr{V}
$$

with the non-proper multiplier

$$
\Pi_\lambda^{\mathrm{Popov}}(s) := \begin{pmatrix} 0 & \operatorname{diag}(\lambda)s \\ -\operatorname{diag}(\lambda)s & 0 \end{pmatrix}.
$$

Note that neither $\Sigma_\gamma^{\mathrm{Popov}}$ (as a map) nor $\Pi_\lambda^{\mathrm{Popov}}$ (as a function on the imaginary axis) are bounded. Still it is not difficult to show (7.4.27). Indeed with $\gamma_{AB} := \|(sI - A)^{-1}B\|_\infty$ the response of (7.4.41) satisfies

$$
\|\dot{z}\| \le \|CA\|\|\xi\| + \|CB\|\|u\| + \|CA\|\|v\| \le (\gamma_{AB}\|CA\| + \|CB\|)\|u\| + \|CA\|\|v\|.
$$

Therefore

$$\Sigma_\lambda^{\text{Popov}} \begin{pmatrix} w \\ Mw + Mu + Nv \end{pmatrix} - \Sigma_\lambda^{\text{Popov}} \begin{pmatrix} w \\ Mw \end{pmatrix} = \Sigma_\lambda^{\text{Popov}} \begin{pmatrix} w \\ Mu + Nv \end{pmatrix} =$$

$$= \int_0^\infty w(t)^T \text{diag}(\lambda)\dot{z}(t)\,dt \leq \|\text{diag}(\lambda)\|\|w\| \left[(\gamma_{AB}\|CA\| + \|CB\|)\|u\| + \|CA\|\|v\| \right]$$

and the right-hand side is a quadratic form in $\|w\|$, $\|u\|$, $\|v\|$ not depending on $\|w\|^2$.

Let us now verify (7.4.29). With $z = Mu + Nv \in M\mathcal{L}_2^k + N\mathcal{V}$ and $\tau \in [0,1]$ we infer

$$\Sigma_\lambda^{\text{Popov}} \begin{pmatrix} \tau\Delta(z) \\ z \end{pmatrix} = \sum_{v=1}^k \int_0^\infty \tau\lambda_v\phi_v(z_v(t))\dot{z}_v(t)\mathrm{d}t =$$

$$= \sum_{v=1}^k \tau\lambda_v \int_0^\infty \frac{d}{dt} I_{\phi_v}(z_v(t))\,dt = -\sum_{v=1}^k \tau\lambda_v I_{\phi_v}(z_v(0)).$$

If we recall $|\phi_v(x)| \leq \max\{|\alpha_v|,|\beta_v|\}|x|$ and hence $|I_{\phi_v}(x)| \leq \frac{1}{2}\max\{|\alpha_v|,|\beta_v|\}|x|^2$ as well as $z_v(0) = C_v v(0)$ (where C_v denotes the v-th row of C) we conclude

$$\Sigma_\lambda^{\text{Popov}} \begin{pmatrix} \tau\Delta(z) \\ z \end{pmatrix} \geq -\delta_0\|v(0)\|^2 \quad \text{with} \quad \delta_0 := \sum_{v=1}^k \frac{1}{2}|\lambda_v|\max\{|\alpha_v|,|\beta_v|\}\|C_v\|^2.$$

Let us finally consider (7.4.28). If $w \in \mathcal{L}_2^k$ and $z = Mw$ we clearly have $\hat{z} = T\hat{w}$ and hence

$$\Sigma_\lambda^{\text{Popov}} \begin{pmatrix} w \\ Mw \end{pmatrix} = \int_{-\infty}^\infty \widehat{w}(i\omega)^* \begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \Pi_\lambda^{\text{Popov}}(i\omega) \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} \widehat{w}(i\omega)\mathrm{d}\omega.$$

Since T is strictly proper, (7.4.28) is equivalent to the FDI

$$\begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \Pi_\lambda^{\text{Popov}}(i\omega) \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} \prec 0 \quad \text{for all} \quad \omega \in \mathbb{R} \cup \{\infty\}. \tag{7.4.42}$$

Whenever there exists some $\lambda \in \mathbb{R}^k$ which satisfies this FDI, Theorem 7.35 allows to infer the existence of some $\gamma > 0$ such that, together with (7.3.14),

$$\|z\|^2 \leq \gamma^2\|u\|^2 + \gamma\|v\|^2 + \gamma\delta_0(v) \leq \gamma^2\|u\|^2 + (\gamma\gamma_A^2 + \gamma\delta_0)\|x_0\|^2.$$

This is the stability property which we intended to show.

Clearly (7.4.42) is a generalization of the classical Popov criterion as considered in the motivating introduction to multiple nonlinearities. Since T is strictly proper, the FDI is equivalent to an LMI constraint on $\lambda \in \mathbb{R}^k$. It is a crucial observation that we can easily combine these Popov multipliers with those obtained earlier, such as the full block-multipliers (7.3.19) introduced in the discussion of the circle criterion. Indeed with $\Pi \in \mathbf{\Pi}$ and $\lambda \in \mathbb{R}^k$ we infer that (7.4.29) persists to hold for

$$\Sigma = \Sigma_\Pi + \Sigma_\lambda^{\text{Popov}} \quad \text{with multiplier} \quad \Pi + \Pi_\lambda^{\text{Popov}}(i\omega)$$

a	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8
b_{cir}	2.19	1.6	1.24	1	0.83	0.705	0.611	0.537
b_{popc}	2.19	1.6	1.24	1	0.83	0.705	0.611	0.537
b_{pop}	2.99	2.27	1.8	1.48	1.24	1.06	0.918	0.802
b_{zfc}	3.12	2.21	1.67	1.32	1.08	0.905	0.776	0.677
b_{com}	3.43	2.47	1.9	1.52	1.26	1.06	0.924	0.803
$b_{\text{zf}} = b_{\text{par}}$	4.55	2.91	2.1	1.62	1.31	1.09	0.925	0.803

Table 7.3: Computed stability margins in Example 7.38.

which involves the FDI

$$
\begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \left[\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} + \begin{pmatrix} 0 & \text{diag}(\lambda)(i\omega) \\ (i\omega)^* \text{diag}(\lambda) & 0 \end{pmatrix} \right] \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} =
$$

$$
= \begin{pmatrix} I \\ T(i\omega) \\ (i\omega)T(i\omega) \end{pmatrix}^* \begin{pmatrix} Q & S & \text{diag}(\lambda) \\ S^T & R & 0 \\ \text{diag}(\lambda) & 0 & 0 \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \\ (i\omega)T(i\omega) \end{pmatrix} \prec 0 \text{ for all } \omega \in \mathbb{R} \cup \{\infty\}
$$

and hence the LMI

$$
\begin{pmatrix} A^T X + XA & XB \\ B^T X & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & 0 \\ CA & CB \end{pmatrix}^T \begin{pmatrix} Q & S & \text{diag}(\lambda) \\ S^T & R & 0 \\ \text{diag}(\lambda) & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ C & 0 \\ CA & CB \end{pmatrix} \prec 0. \tag{7.4.43}
$$

Corollary 7.37 (Full Block Multiplier Popov Criterion) *Suppose there exist $\Pi \in \mathbf{\Pi}$, $\lambda \in \mathbb{R}^k$ and $X = X^T$ satisfying (7.4.43). Then there exist constants γ and γ_0 with (7.4.40).*

One can surely specialize Π to the smaller class of multipliers in Example 7.15 in order to reduce computational complexity while possible introducing conservatism. It is stressed that the Popov multipliers should be always applied in combination with those from the circle criterion in order to avoid overly conservative tests.

Example 7.38 Let us continue Examples 7.20 and 7.29. Table 7.3 displays the stability margins b_{popc} and b_{pop} for the classical and the more general Popov criterion. For reasons of comparison we display as well b_{com} for a combination of the general Popov criterion with causal Zames-Falb multipliers. The examples nicely illustrate the expected relations $b_{\text{cir}} \leq b_{\text{popc}} \leq b_{\text{pop}} \leq b_{\text{com}} \leq b_{\text{par}}$ and $b_{\text{cir}} \leq b_{\text{zfc}} \leq b_{\text{zf}} \leq b_{\text{par}}$ as well as $b_{\text{zfc}} \leq b_{\text{com}}$, while the Zames-Falb tests with fixed basis functions is not comparable to the Popov test. For this example, it is interesting to observe that the classical Popov criterion does not lead to an improvement over the circle criterion, while a negative Popov multiplier allows to prove stability for a significantly larger parameter b. Moreover, the combination of Popov and causal Zames-Falb multipliers beats the pure version and is beaten by a Zames-Falb analysis with a simple non-causal multiplier.

7.4.5   Extension to incremental stability and well-posedness

Let us consider assume that M, N, Σ, $\mathscr{V}_e$ and $\mathscr{V}$ have the same properties as described in Section 7.6, while

$$\Delta : \mathscr{L}_{2e}^l \to \mathscr{L}_{2e}^k \quad \text{is causal and incrementally bounded.}$$

In particular for some nonlinear Δ it is relevant to analyze interconnection stability "around a given trajectory". If (7.4.26) is well-posed and $(u_0, v_0) \in \mathscr{L}_{2e}^k \times \mathscr{V}$ is an external disturbance let us denote the nominal response by $z_0 = S(u_0, v_0)$. For $(u, v) \in \mathscr{L}_2^k \times \mathscr{V}$ and $z = S(u_0 + u, v_0 + v)$ it is then of interest to bound the norm of the increment

$$\bar{z} := z - z_0 = S(u_0 + u, v_0 + v) - S(u_0, v_0)$$

in terms of those of the disturbance increment (u, v). With $w_0 = \Delta(z_0)$, we note for this purpose that $z_0 = Mw_0 + Mu_0 + Nv_0$ and $z = M\Delta(z) + M(u_0 + u) + N(v_0 + v)$ and hence

$$\bar{z} = M(\Delta(z_0 + \bar{z}) - \Delta(z_0)) + Mu + Mv.$$

This can in turn be interpreted as

$$\bar{z} = M\bar{w} + Mu + Nv \quad \text{interconnected with} \quad \bar{w} = \Delta(z_0 + \bar{z}) - \Delta(z_0).$$

Therefore we can directly apply Theorem 7.35 with $\Delta(\cdot)$ replaced by $\Delta(z_0 + \cdot) - \Delta(z_0)$ in order to guarantee incremental stability. Since Δ is causal and incrementally bounded, let us stress that $\Delta(z_0 + \cdot) - \Delta(z_0)$ is causal and bounded such that Theorem 7.35 leads to the following incremental quadratic stability result.

Corollary 7.39 *Suppose that Δ is causal and incrementally bounded and that $(u_0, v_0) \in \mathscr{L}_{2e}^k \times \mathscr{V}_e$. If there exists $\varepsilon > 0$ and $\delta_0 : \mathscr{V} \to \mathbb{R}$ with (7.4.28) and*

$$\Sigma \left(\begin{array}{c} \tau\Delta(z_0 + Mu + Nv) - \tau\Delta(z_0) \\ Mu + Nv \end{array} \right) \geq -\delta_0(v) \text{ for all } \tau \in [0, 1], (u, v) \in \mathscr{L}_2^k \times \mathscr{V}$$

then there exists some $\gamma > 0$ such that

$$\|S(u_0 + v, v_0 + v) - S(u_0, v_0)\|^2 \leq \gamma^2(\|u\|^2 + \|v\|^2) + \gamma\delta_0(v) \text{ for all } (u, v) \in \mathscr{L}_2^k \times \mathscr{V}.$$

This result involves an incremental version of the quadratic constrain on Δ, without any modification of the quadratic constraint on M. The main goal of this section is to show that validity of this incremental quadratic constraint for **all** $(u_0, v_0) \in \mathscr{L}_{2e}^k \times \mathscr{V}_e$ does also imply well-posedness. We can hence get rid of the **assumption** that the interconnection (7.4.26) is well-posed and actually **conclude** it as a result. We require the mild extra hypothesis that $0 \in \mathscr{V}$ which is of course true in the important case that $\mathscr{V}_e$ is a subspace of $\mathscr{L}_{2e}^m$.

Theorem 7.40 *Suppose that*

- *there exist $\varepsilon > 0$ such that*

$$\Sigma \begin{pmatrix} w \\ Mw \end{pmatrix} \leq -\varepsilon \|w\|^2 \ \text{ for all } \ w \in \mathscr{L}_2^k; \tag{7.4.44}$$

- $0 \in \mathscr{V}$ *and there exists* $\delta_0 : \mathscr{V} \to \mathbb{R}$ *with* $\delta_0(0) = 0$ *and such that*

$$\Sigma \begin{pmatrix} \tau\Delta(z + Mu + Nv) - \tau\Delta(z) \\ Mu + Nv \end{pmatrix} \geq -\delta_0(v)$$
$$\text{for all } \ \tau \in [0,1], \ u \in \mathscr{L}_2^k, \ v \in \mathscr{V}, \ z \in M\mathscr{L}_{2e}^k + N\mathscr{V}_e. \tag{7.4.45}$$

Then the feedback system (7.4.26) is well-posed. Moreover there exists some $\gamma > 0$ such that the response map S satisfies

$$\|S(u_0 + u, v_0 + v) - S(u_0, v_0)\|^2 \leq \gamma^2(\|u\|^2 + \|v\|^2) + \gamma\delta_0(v)$$
$$\text{for all } \ (u,v) \in \mathscr{L}_2^k \times \mathscr{V}, \ (u_0, v_0) \in \mathscr{L}_{2e}^k \times \mathscr{V}_e. \tag{7.4.46}$$

Proof. We use again the abbreviations $\mathscr{D}_e := \mathscr{L}_{2e}^k \times \mathscr{V}_e$ and $\mathscr{D} := \mathscr{L}_2^k \times \mathscr{V}$, and with $\tau \in [0,1]$ we consider (7.4.26) for $\tau\Delta$ replacing Δ. Whenever the corresponding interconnection is well-posed, the response map $\mathscr{D}_e \ni (u,v) \to z \in \mathscr{L}_{2e}^l$ is denoted by S_τ. For ease of exposition we express well-posedness by just saying that S_τ exists. As a means of reference for the proof of the following three steps let us finally introduce the property

$$S_\tau \text{ exists and satisfies } \ S_\tau((u_0, v_0) + \mathscr{D}) - S_\tau(u_0, v_0) \subset \mathscr{L}_2^l \ \text{ for all } \ (u_0, v_0) \in \mathscr{D}_e. \tag{7.4.47}$$

Step 1. There exists some (τ-independent) γ such that (7.4.47) for $\tau \in [0,1]$ implies

$$\|S_\tau(u_0 + u, v_0 + v) - S_\tau(u_0, v_0)\|^2 \leq \gamma^2 (\|u\|^2 + \|v\|^2) + \gamma\delta_0(v)$$
$$\text{for all } \ (u,v) \in \mathscr{D}, \ (u_0, v_0) \in \mathscr{D}_e. \tag{7.4.48}$$

Indeed, in view of unchanged hypotheses on M and Σ one can still conclude as in Step 1 of the proof of Theorem 7.35 that there exists some $\gamma > 0$ with (7.4.33) for all $w \in \mathscr{L}_2^k$, $(u,v) \in \mathscr{D}$. If $w, w_0 \in \mathscr{L}_{2e}^l$ satisfy $w - w_0 \in \mathscr{L}_2^k$ we infer for all $(u,v) \in \mathscr{D}$ that

$$\Sigma \begin{pmatrix} w - w_0 \\ M(w - w_0) + Mu + Nv \end{pmatrix} +$$
$$+ \frac{1}{\gamma}\|M(w - w_0) + Mu + Nv\|^2 - \gamma(\|u\|^2 + \|v\|^2) \leq 0. \tag{7.4.49}$$

For $(u_0, v_0) \in \mathscr{D}_e$ and $(u,v) \in \mathscr{D}$ set $z_0 := S_\tau(u_0, v_0)$ and $z := S_\tau(u_0 + u, v_0 + v)$. Then $z - z_0 \in \mathscr{L}_2^l$ by (7.4.47). Hence $w_0 := \tau\Delta(z_0)$ and $w := \tau\Delta(z)$ satisfy $w - w_0 \in \mathscr{L}_2^k$ since Δ has finite incremental

gain. Due to $z_0 = Mw_0 + Mu_0 + Nv_0$ and $z = Mw + M(u_0 + u) + N(v_0 + v)$ we note that $z - z_0 = M(w - w_0) + Mu + Nv$. Therefore (7.4.49) implies

$$\Sigma \begin{pmatrix} \tau\Delta(z) - \tau\Delta(z_0) \\ z - z_0 \end{pmatrix} + \frac{1}{\gamma}\|S_\tau(u_0 + u, v_0 + v) - S_\tau(u_0, v_0)\|^2 - \gamma(\|u\|^2 + \|v\|^2) \le 0.$$

On the other hand, because $z_0 = M(w_0 + u_0) + Nv_0 \in M\mathscr{L}_{2e}^k + N\mathscr{V}_e$ and $z = z_0 + M\tilde{u} + Nv$ with $\tilde{u} = w - w_0 + u \in \mathscr{L}_2^k$, $v \in \mathscr{V}$, we can apply (7.4.45) in order to arrive at (7.4.48).

Step 2. With γ from Step 1 choose any $\rho_0 > 0$ for which $\rho_0 < 1/(\gamma\|\Delta\|_i)$. Then

(7.4.47) holds for $\tau \in [0,1]$, $\tau + \rho \in [0,1]$, $|\rho| \le \rho_0$

$$\implies \quad (7.4.47) \text{ holds for } \tau + \rho \text{ replacing } \tau. \quad (7.4.50)$$

Fix ρ and τ as in (7.4.50). Existence of $S_{\tau+\rho}$ involves solving $z - (\tau + \rho)M\Delta(z) = Mu + Nv$ for $z \in \mathscr{L}_{2e}^l$. This equation can be rewritten as $z - \tau M\Delta(z) = M(\rho\Delta(z) + u) + Nv$ or as $z = S_\tau(\rho\Delta(z) + u, v)$ since S_τ is assumed to exist. This motivates to introduce the map

$$F_{(u,v)} : \mathscr{L}_{2e}^l \to \mathscr{L}_{2e}^l, \quad F_{(u,v)}(z) := S_\tau(\rho\Delta(z) + u, v) \text{ for any } (u,v) \in \mathscr{D}_e.$$

As just seen we are lead to the following crucial equivalence for $(u,v) \in \mathscr{D}_e$ and $z \in \mathscr{L}_{2e}^l$:

$$z - (\tau + \rho)M\Delta(z) = Mu + Nv \text{ iff } z = F_{(u,v)}(z). \quad (7.4.51)$$

Moreover, if exploiting causality of Δ and of S_τ in the first argument, it is easy to check that $F_{(u,v)}$ is causal; indeed for all $T > 0$ we get

$$F_{(u,v)}(z)_T = S_\tau(\rho\Delta(z) + u, v)_T = S_\tau(\rho\Delta(z)_T + u_T, v)_T =$$
$$= S_\tau(\rho\Delta(z_T)_T + u_T, v)_T = S_\tau(\rho\Delta(z_T) + u, v)_T = F_{(u,v)}(z_T)_T.$$

Due to $(\rho\Delta(z) + u)_T = (\rho\Delta(z) + u_T)_T = (\rho\Delta(z_T) + u)_T$ we actually have

$$F_{(u,v)}(z)_T = F_{(u_T, v)}(z)_T = F_{(u,v)}(z_T)_T. \quad (7.4.52)$$

Now choose $(u_0, v_0) \in \mathscr{D}_e$ and $(u,v) \in \mathscr{D}$ as well as $z_1, z_2 \in \mathscr{L}_{2e}^l$. For $T > 0$ we infer from $(\rho\Delta(z_1)_T + (u_0)_T, v_0) \in \mathscr{D}_e$ and $(\rho\Delta(z_2)_T - \rho\Delta(z_1)_T + u_T, v) \in \mathscr{D}$ together with the assumption (7.4.47) that

$$S_\tau(\rho\Delta(z_2)_T + (u_0 + u)_T, v_0 + v) - S_\tau(\rho\Delta(z_1)_T + (u_0)_T, v_0) \in \mathscr{L}_2^l.$$

If again exploiting causality and with (7.4.48) we get

$$\|F_{(u_0+u, v_o+v)}(z_2)_T - F_{(u_0, v_0)}(z_1)_T\| =$$
$$= \|S_\tau(\rho\Delta(z_2) + u_0 + u, v_0 + v)_T - S_\tau(\rho\Delta(z_1) + u_0, v_0)_T\| =$$
$$= \|[S_\tau(\rho\Delta(z_2)_T + (u_0 + u)_T, v_0 + v) - S_\tau(\rho\Delta(z_1)_T + (u_0)_T, v_0)]_T\| \le$$
$$\le \|S_\tau(\rho\Delta(z_2)_T + (u_0 + u)_T, v_0 + v) - S_\tau(\rho\Delta(z_1)_T + (u_0)_T, v_0)\| \le$$
$$\le \gamma\|\rho\Delta(z_2)_T - \rho\Delta(z_1)_T + u_T\| + \gamma\|v\| + \sqrt{\gamma\delta_0(v)} \le$$
$$\le \gamma\|\rho\Delta(z_1)_T - \rho\Delta(z_1)_T\| + \gamma\|u_T\| + \gamma\|v\| + \sqrt{\gamma\delta_0(v)} \le$$
$$\le \gamma\rho_0\|\Delta\|_i\|(z_2)_T - (z_1)_T\| + \gamma\|u\| + \gamma\|v\| + \sqrt{\gamma\delta_0(v)}. \quad (7.4.53)$$

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After having collected all required properties of the map $F_{(u,v)}$, let us now fix $(u_0, v_0) \in \mathscr{D}_e$ and $T_1 > 0$. If we recall $\gamma \rho_0 \|\Delta\|_i < 1$, we infer from (7.4.53) for $(u,v) = (0,0)$ and with $\delta_0(0) = 0$ that the map $z \to F_{(u_0,v_0)}(z)_{T_1}$ is a strict contraction on the the space $(\mathscr{L}_{2e}^l)_{T_1}$ which can be identified with $\mathscr{L}_2^l[0, T_1]$ and is hence complete. By Banach's fixed-point theorem there exists a unique $z_1 \in (\mathscr{L}_{2e}^l)_{T_1}$ with $F_{(u_0,v_0)}(z_1)_{T_1} = z_1$.

For any diverging sequence $T_1 < T_2 < \cdots$ denote the unique fixed-point corresponding to T_ν by z_ν. Note that $z_\nu = (z_\nu)_{T_\nu}$. If $\nu < \mu$ we also infer with $T_\nu < T_\mu$ and causality that

$$F_{(u_0,v_0)}((z_\mu)_{T_\nu})_{T_\nu} = F_{(u,v)}(z_\mu)_{T_\nu} = [F_{(u_0,v_0)}(z_\mu)_{T_\mu}]_{T_\nu} = (z_\mu)_{T_\nu}.$$

Due to uniqueness of z_ν as a fixed point, we can conclude

$$(z_\mu)_{T_\nu} = z_\nu \quad \text{for} \quad \nu \le \mu.$$

A little reflection shows that the signal $\tilde{z} \in \mathscr{L}_{2e}^l$ is hence well-defined by the assignment

$$\tilde{z}_{T_\nu} = z_\nu \quad \text{for} \quad \nu = 1, 2, \ldots.$$

Based on this construction we are now ready to prove existence of $S_{\tau+\rho}$, which means that (7.4.26) is well-posed for $(\tau + \rho)\Delta$ replacing Δ.

- **Existence of response.** Again by causality,

  $$F_{(u_0,v_0)}(\tilde{z})_{T_\nu} = F_{(u_0,v_0)}(\tilde{z}_{T_\nu})_{T_\nu} = F_{(u_0,v_0)}(z_\nu)_{T_\nu} = z_\nu = (z_\nu)_{T_\nu} = \tilde{z}_{T_\nu}$$

  for all $\nu = 1, 2, \ldots$. This implies $F_{(u_0,v_0)}(\tilde{z}) = \tilde{z}$ and hence $\tilde{z} - (\tau + \rho)M\Delta(\tilde{z}) = Mu_0 + Nv_0$ by (7.4.51).

- **Uniqueness of response.** If $\hat{z} \in \mathscr{L}_{2e}^l$ satisfies $\hat{z} - (\tau + \rho)M\Delta(\hat{z}) = Mu_0 + Nv_0$ we have $\hat{z} = F_{(u_0,v_0)}(\hat{z})$ and thus $\hat{z}_{T_\nu} = F_{(u_0,v_0)}(\hat{z})_{T_\nu} = F_{(u_0,v_0)}(\hat{z}_{T_\nu})_{T_\nu}$ and thus $(\hat{z})_{T_\nu} = z_\nu = \tilde{z}_{T_\nu}$ due to fixed-point uniqueness. Since ν was arbitrary we infer $\hat{z} = \tilde{z}$.

- **Definition of response map.** We have just shown that $\tilde{z}$ is the unique feedback response to $(u_0, v_0) \in \mathscr{D}_e$. Since $(u_0, v_0) \in \mathscr{D}_e$ was arbitrary, we conclude that the feedback response map $S_{\tau+\rho}$ is well-defined.

- **Causality.** It remains to show that $S_{\tau+\rho}$ is causal in the first argument. Let $T > 0$ and $\hat{z} = S_{\tau+\rho}((u_0)_T, v_0)$. Then $\hat{z} = F_{((u_0)_T, v_0)}(\hat{z})$ and hence with (7.4.52) we have

  $$\hat{z}_T = F_{((u_0)_T, v_0)}(\hat{z})_T = F_{(u_0,v_0)}(\hat{z}_T)_T \quad \text{an clearly also} \quad \tilde{z}_T = F_{(u_0,v_0)}(\tilde{z}_T)_T.$$

  Fixed-point uniqueness implies $\hat{z}_T = \tilde{z}_T$ and thus $S_{\tau+\rho}((u_0)_T, v_0)_T = S_{\tau+\rho}(u_0, v_0)_T$.

Let us now finish the proof of Step 2. For $(u_0, v_0) \in \mathscr{D}_e$ and $(u, v) \in \mathscr{D}$ set $z_0 = S_{\tau+\rho}(u_0, v_0)$ and $z = S_{\tau+\rho}(u_0 + u, v_0 + v)$ which are both in $\mathscr{L}_{2e}^l$. With (7.4.53) we obtain

$$(1 - \gamma \rho_0 \|\Delta\|_i) \|(z_1)_T - (z_2)_T\| \le \gamma \|u\| + \gamma \|v\| + \sqrt{\gamma \delta_0(v)}$$

for all $T > 0$. Since the right-hand side is bounded for $T \to \infty$ and since $(1 - \gamma \rho_0 \|\Delta\|_i) > 0$, we arrive at $z_1 - z_2 \in \mathscr{L}_2^l$.

Step 3. Clearly (7.4.47) holds for $\tau = 0$. Since ρ_0 in Step 2 does not depend on τ, we can inductively apply (7.4.50) in order to infer that (7.4.47) persists to hold for $\tau \in [0, \nu \rho_0] \cap [0, 1]$ and all $\nu = 1, 2, \dots$, and thus also for $\tau = 1$. Then (7.4.48) leads to (7.4.46). ∎

The following easy-to-prove variant allows to draw conclusions about causality of the response map in the input $v \in \mathscr{V}_e$.

Corollary 7.41 *In addition to all hypotheses in Theorem 7.40 suppose that N is causal and that* $\mathscr{V} = \mathscr{L}_{2e}^m$. *Then the response S is causal.*

Proof. We only point out the minor variations that are required in the proof of Theorem 7.40. Existence of S_τ involves causality in both arguments. In Step 2 one can then reason as for (7.4.52) that $F_{u,v}(z)_T = F_{u,v}(z)_T$. This allows to to apply the same arguments as in the proof in order to show that $S_{\tau+\rho}$ is causal in the second argument, which in turn implies causality of $S_{\tau+\rho}$. We exploit causality of N in order to guarantee that S_0 is causal in Step 3. ∎

Remark 7.42

- Even if considering nonlinear systems Δ they often are guaranteed to have the property $\Delta(0) = 0$. If we can apply Theorem 7.40, we infer that $(0,0) \in \mathscr{D}$ has the response $z = 0$ and hence, due to well-posedness, $S(0,0) = 0$. Then (7.3.23) boils down to the conclusion of Theorem 7.35 and the feedback interconnection is also stable.

- If Δ is affine or even linear, the hypothesis of Theorem 7.40 are identical to those of Theorem 7.35. Then the latter indeed guarantees well-posedness, stability and incremental stability of the feedback interconnection at once. If Δ might be infinite-dimensional (such as systems described by partial differential equations or a delay in continuous-time) it could be very beneficial that it is not required to verify well-posedness at the outset.

- We have made use of a rather straightforward application of the global version of Banach's fixed-point theorem. As variants on could work with well-established local versions thereof. We stress again that the presented technique of proof allows the derivation of variants that are not covered by Theorem 7.40.

7.5 Robust stability analysis

7.6 Integral quadratic constraints

7.6.1 Stability tests with integral quadratic constraints

In this section we assume that the uncertainties Δ are general but that $z = Mw$ is defined with a proper stable rational matrix $\widehat{M}$ as

$$\widehat{z}(i\omega) = \widehat{M}(i\omega)\widehat{w}(i\omega), \ \ i\omega \in \mathbb{C}^0.$$

Recall that $\widehat{x}$ denotes the Fourier transform of the signal $x \in \mathscr{L}_2$.

Instead of general quadratically continuous mappings, let us consider so-called integral quadratic forms. Suppose $\Pi : i\omega \to \Pi(i\omega)$ is any (measurable) mapping that assigns to every $i\omega \in \mathbb{C}^0$ a Hermitian matrix $\Pi(i\omega)$ of dimension $(k+l) \times (k+l)$ that is bounded:

$$\|\Pi(i\omega)\| \leq p \ \text{ for all } \ i\omega \in \mathbb{C}^0.$$

(Note that we will consider in most cases mappings that are defined with a rational matrix valued function $\Pi(s)$; then it is just required that this rational matrix is Hermitian on the imaginary axis, and that it has neither a pole in $\mathbb{C}^0$ nor at infinity such that it is proper.)

For any $x, y \in \mathscr{L}_2^{k+l}$ we can define with their Fourier transforms $\widehat{x}, \widehat{y}$ the mapping

$$\langle x, y \rangle := \int_{-\infty}^{\infty} \widehat{x}(i\omega)^* \Pi(i\omega)\widehat{y}(i\omega)\mathrm{d}\,\omega$$

which satisfies (**??**) with bound $\sigma = p$.

Condition (**??**) then amounts to

$$\int_{-\infty}^{\infty} \widehat{w}(i\omega)^* \left(\begin{array}{c} I \\ \widehat{M}(i\omega) \end{array} \right)^* \Pi(i\omega) \left(\begin{array}{c} I \\ \widehat{M}(i\omega) \end{array} \right) \widehat{w}(i\omega)\, d\omega \leq -\frac{\varepsilon}{2\pi} \int_{-\infty}^{\infty} \widehat{w}(i\omega)^* \widehat{w}(i\omega)\mathrm{d}\,\omega$$

for all $w \in \mathscr{L}_2^k$. This is obviously implied by the frequency domain inequality (FDI)

$$\left(\begin{array}{c} I \\ \widehat{M}(i\omega) \end{array} \right)^* \Pi(i\omega) \left(\begin{array}{c} I \\ \widehat{M}(i\omega) \end{array} \right) \leq -\frac{\varepsilon}{2\pi}I \ \text{ for all } \ i\omega \in \mathbb{C}^0.$$

It is not required for our arguments and not difficult to see that the converse holds as well; both characterization are in fact equivalent.

Lemma 7.43 *Suppose Π is a (measurable) bounded Hermitian valued mapping on $\mathbb{C}^0$. Then the following two statements are equivalent:*

- $\int_{-\infty}^{\infty} \widehat{x}(i\omega)^* \Pi(i\omega)\widehat{x}(i\omega)\, d\omega \leq -\alpha \int_{-\infty}^{\infty} \widehat{x}(i\omega)^* \widehat{x}(i\omega)\, d\omega$ *for all* $x \in \mathscr{L}_2$.

- $\Pi(i\omega) \leq -\alpha I$ *for all* $i\omega \in \mathbb{C}^0$.

The reason for this re-formulation: the frequency domain inequality is easier to check.

Now we get as immediate corollaries to Theorems **??** and **??** the following stability results using *integral quadratic constraints* (IQC's).

Theorem 7.44 *Suppose that, for all* $\Delta \in \mathbf{\Delta}$, $\mathscr{I}_M(\Delta)$ *is well-posed and that*

$$\int_{-\infty}^{\infty} \left(\begin{array}{c} \widehat{\Delta(z)}(i\omega) \\ \widehat{z}(i\omega) \end{array} \right)^* \Pi(i\omega) \left(\begin{array}{c} \widehat{\Delta(z)}(i\omega) \\ \widehat{z}(i\omega) \end{array} \right) d\omega \geq 0 \ \text{ for all } \ z \in \mathscr{L}_2^l. \tag{7.6.1}$$

If there exists an $\varepsilon > 0$ *with*

$$\left(\begin{array}{c} I \\ \widehat{M}(i\omega) \end{array} \right)^* \Pi(i\omega) \left(\begin{array}{c} I \\ \widehat{M}(i\omega) \end{array} \right) \leq -\varepsilon I \ \text{ for all } \ \omega \in \mathbb{R}, \tag{7.6.2}$$

then the $\mathscr{L}_2$-*gain of* $\mathscr{I}_M^{-1}(\Delta)$ *is bounded uniformly in* $\Delta \in \mathbf{\Delta}$.

Theorem 7.45 *Suppose that any* $\Delta \in \mathbf{\Delta}$ *has finite incremental* $\mathscr{L}_2$-*gain and satisfies*

$$\int_{-\infty}^{\infty} \left(\begin{array}{c} \widehat{\Delta(z_1)}(i\omega) - \widehat{\Delta(z_2)}(i\omega) \\ \widehat{z_1}(i\omega) - \widehat{z_2}(i\omega) \end{array} \right)^* \Pi(i\omega) \left(\begin{array}{c} \widehat{\Delta(z_1)}(i\omega) - \widehat{\Delta(z_2)}(i\omega) \\ \widehat{z_1}(i\omega) - \widehat{z_2}(i\omega) \end{array} \right) d\omega \geq 0 \tag{7.6.3}$$

for all $z_1, z_2 \in \mathscr{L}_2^l$. *If there exists an* $\varepsilon > 0$ *with* (7.6.2), *then* $\mathscr{I}_M(\Delta)$ *is well-posed, and the incremental* $\mathscr{L}_2$-*gain of its inverse is uniformly bounded in* $\Delta \in \mathbf{\Delta}$.

Remark 7.46 • One should read $\widehat{\Delta(z)}(i\omega)$ correctly: Take z, let it pass through Δ to get the signal $\Delta(z)$, take its Fourier transform $\widehat{\Delta(z)}$, and evaluate this Fourier transform at $i\omega$ to obtain $\widehat{\Delta(z)}(i\omega)$. Therefore, the signal z with power distribution $\widehat{z}$ is mapped into the signal $\Delta(z)$ with power distribution $\widehat{\Delta(z)}$. (In general, of course, there is no nice operation - such as the multiplication by a transfer matrix - that maps $\widehat{z}$ directly into $\widehat{\Delta(z)}$. However, since we only transform signals, no complication arises).

The inequality (7.6.1) defined via Π hence restricts how the power distribution of z can and cannot be rearranged in $\widehat{\Delta(z)}$; (7.6.1) could be called a power distribution constraint. The constraint (7.6.3) admits the same interpretation for increments.

- In principal, the inequality (7.6.2) is easy to verify: one just needs to plot the largest eigenvalue of the left-hand side over frequency and read of the maximum that this curve takes. This could be viewed as a generalization of plotting the largest singular values of a certain transfer matrix to apply the small-gain theorem.

  If Π is real-rational and proper, the Kalman-Yakubovich-Popov Lemma allows to reduce this condition to the solvability of a linear matrix inequality; this is the reason why IQC's play such prominent role in the LMI approach to robust control. We will elaborate on these points in Section **??**.

7.6.2 The philosophy for applying IQC's

So far we have considered *one* quadratically continuous mapping Σ and *one* IQC to characterize stability. For small-gain and passivity conditions, this was sufficient to arrive at standard stability results. However, if one has a more detailed picture about the uncertainty, one can often find more than one IQC that are satisfied by the uncertainties.

For the purpose of illustration let us look at a simple example. Consider the structured nonlinear uncertainties $\Delta : \mathscr{L}_{2e}^l \to \mathscr{L}_{2e}^k$ that are defined for fixed partitions

$$
z = \begin{pmatrix} z^1 \\ \vdots \\ z^m \end{pmatrix}, \quad w = \begin{pmatrix} w^1 \\ \vdots \\ w^m \end{pmatrix}
$$

(where the signals z^j and w^j can have different sizes) with the causal mappings $\Delta_j : \mathscr{L}_{2e} \to \mathscr{L}_{2e}$, $\Delta_j(0) = 0$, as

$$
w = \Delta(z), \quad \Delta(\begin{pmatrix} z^1 \\ \vdots \\ z^m \end{pmatrix}) = \begin{pmatrix} \Delta_1(z^1) \\ \vdots \\ \Delta_m(z^m) \end{pmatrix}.
$$

Furthermore, it is assumed that $\|\Delta_j\|_{2i} \leq 1$ such that, as well, $\|\Delta_j\|_2 \leq 1$.

Note that the set of all these uncertainties is star-shaped. Due to $\|\Delta\|_{2i} \leq 1$, the incremental small-gain theorem applies. Then $\|\widehat{M}\|_\infty < 1$ implies that $\mathscr{I}_M(\Delta)^{-1}$ exists, is causal, and uniformly incrementally bounded. However, this also holds for the much larger class of all uncertainties Δ with $\|\Delta\|_{2i} \leq 1$, even if they do not have the specific structure considered here.

Hence we should find more IQC's that provide a way to capture this structure. Motivated by μ-theory, we consider the IQC's defined with the constant matrices Π given as

$$
\Pi = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}, \; S = 0, \; Q = \text{diag}(-r_1 I, \ldots, -r_m I), \; R = \text{diag}(r_1 I, \ldots, r_m I), \; r_j > 0 \quad (7.6.4)
$$

where the sizes of the identity blocks in Q and R correspond to the sizes of the signals w^j and z^j respectively. We infer

$$
\int_0^\infty \begin{pmatrix} \Delta(z_1)(t) - \Delta(z_2)(t) \\ z_1(t) - z_2(t) \end{pmatrix}^T \Pi \begin{pmatrix} \Delta(z_1)(t) - \Delta(z_2)(t) \\ z_1(t) - z_2(t) \end{pmatrix} dt =
$$

$$
= \int_0^\infty \sum_{j=1}^m -r_j \|\Delta_j(z_1)(t) - \Delta_j(z_2)(t)\|^2 + r_j \|z_1(t) - z_2(t)\|^2 \, dt =
$$

$$
= \sum_{j=1}^m \int_0^\infty r_j [\|z_1(t) - z_2(t)\|^2 - \|\Delta_j(z_1)(t) - \Delta_j(z_2)(t)\|^2] \, dt \geq 0
$$

such that the incremental IQC's (7.6.3) hold for all uncertainties and for all Π.

We have found a whole family of IQC's for our class of uncertainties, parameterized by the numbers r_j. If we just find among these infinitely many IQC's one for which, in addition, the FDI (7.6.2) holds, we conclude exactly the same stability properties of $\mathscr{I}_M(\Delta)$ as before.

Again, we stress that *all* IQC's (7.6.3) must be satisfied by the uncertainties, but only for *one* IQC we need to assure (7.6.2)! Hence, the more IQC's we find for the uncertainties, the more freedom we have if trying to fulfill the FDI and the better the chances are to verify robust stability.

Let us now have a more detailed look at (7.6.2) for the specific scalings (7.6.4). The inequality simply reads as

$$\widehat{M}(i\omega)^* R \widehat{M}(i\omega) - R < 0 \quad \text{for all} \quad \omega \in \mathbb{R} \cup \{\infty\}. \tag{7.6.5}$$

(Since we have replaced $\leq -\varepsilon I$ by < 0, we have to include $\omega = \infty$ in the condition. Why?) The goal is to find some R (structured as in (7.6.4)) that satisfies this FDI. It will turn out that the search for R can be cast into an LMI problem.

In order to relate to μ-theory, re-parameterize

$$R = D^T D$$

with D in the same class as R. Then $\widehat{M}(i\omega)^* D^T D \widehat{M}(i\omega) - D^T D < 0$ is equivalent to $\|D\widehat{M}(i\omega)D^{-1}\| < 1$ if $\|.\|$ denotes the maximal singular value for complex matrices. Therefore, (7.6.5) is nothing but

$$\|D\widehat{M}D^{-1}\|_\infty < 1 \tag{7.6.6}$$

which is a scaled H_∞ condition. Such conditions - possibly with frequency dependent scalings D - appear in μ-theory. Note, however, that the conclusions made in μ-theory are usually only valid for *linear time-invariant uncertainties that admit a Fourier transform with suitable properties*. Our conclusions hold for a much larger class of uncertainties since our proof was not based on a Nyquist type argument in the frequency domain.

We have shown that we can replace $\|\widehat{M}\|_\infty < 1$ by the scaled small-gain condition (7.6.6) to come to the same robust stability conclusions. The scalings D capture the knowledge about the structure of the uncertainties and provide us extra freedom to satisfy (7.6.5). Hence, the scalings reduce the conservatism that is involved in the simple but rough condition $\|\widehat{M}\|_\infty < 1$.

Let us introduce a terminology: We will call the matrices Π that define the IQC's *scalings* or *multipliers*. The first name is motivated by the above mentioned relation to μ-theory. The second name reminds of the relation to classical multipliers that have been used in loop transformation arguments.

The example reveals the philosophy in applying the robust stability results discussed here: Try to find as many multipliers Π as possible such that the IQC's (7.6.1) (or (7.6.3)) hold for the considered class of uncertainties. Then find, among all these multipliers, one that also satisfies the FDI (7.6.2). If this is possible, one can conclude (existence and) uniform boundedness of the (incremental) $\mathscr{L}_2$-gain of $\mathscr{I}_M(\Delta)^{-1}$.

A simple trick often allows to increase the number of multipliers. Indeed, if $\Pi_1, \ldots, \Pi_k$ are multipli-

ers that satisfy (7.6.1) (or (7.6.3)), the same is true of all

$$\sum_{j=1}^{k} \tau_j \Pi_j \ \text{ if } \ \tau_j \geq 0. \tag{7.6.7}$$

One can hence easily construct out of finitely many multipliers an infinite family of multipliers parameterized by τ_j. The same trick applies to an infinite set of multipliers. (Those familiar with the corresponding concepts will recognize that we just need to take the convex conic hull; any set of multipliers can, therefore, always assumed to be a convex cone.)

Finding multipliers such that a specific class of uncertainties satisfies the corresponding IQC is not really supported by theory; this is indeed the hard part in concrete applications. For suitable parameterizations of the family of considered multipliers (such as (7.6.7) or more general versions), the second step of finding one multiplier that also renders the FDI (7.6.2) satisfied will turn out to be an LMI problem.

7.6.3 Examples of IQC's

In what follows we provide a non-exhaustive list of uncertainties and suitable multipliers. We recall that one needs to always verify Assumption **??**, in particular star-shapeness with center 0, in order to apply Theorem **??**, Theorem **??** or their IQC counterparts.

- The structured nonlinear uncertainties

$$\Delta(\begin{pmatrix} z^1 \\ \vdots \\ z^m \end{pmatrix}) = \begin{pmatrix} \Delta_1(z^1) \\ \vdots \\ \Delta_m(z^m) \end{pmatrix} \tag{7.6.8}$$

with causal Δ_j that satisfy $\|\Delta_j\|_2 \leq 1$ or $\|\Delta_j\|_{2i} \leq 1$ fulfill (7.6.1) or (7.6.3) for the class of multipliers

$$\mathbf{\Pi} := \{ \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix}, \ Q = \text{diag}(-r_1 I, \ldots, -r_m I), R = \text{diag}(r_1 I, \ldots, r_m I) > 0 \}. \tag{7.6.9}$$

- In (7.6.8) we can confine the attention to *linear* causal mappings Δ_j only. Beautiful results by Shamma and Megretsky [29, 52] show that, then, the resulting scaled H_∞-condition (7.6.2) is not only *sufficient* for robust stability (as we have proved) but even *necessary* (what is harder to show).

- We can specialize further and use (7.6.9) also for the block-diagonal time-varying parametric uncertainties

$$w^j(t) = \Delta_j(t)z^j(t)$$

with (measurable) matrix valued functions satisfying

$$\|\Delta_j(t)\| \leq 1 \ \text{ for all } \ t \geq 0.$$

Equivalently, we have

$$w(t) = \Delta(t)z(t), \ \ \Delta(t) = \mathrm{diag}(\Delta_1(t), \dots, \Delta_m(t)), \ \ \|\Delta(t)\| \leq 1 \ \text{ for } \ t \geq 0.$$

In this case, for any Π in the class (7.6.9), the uncertainties even satisfy the quadratic constraint

$$\begin{pmatrix} \Delta(t) \\ I \end{pmatrix}^T \Pi \begin{pmatrix} \Delta(t) \\ I \end{pmatrix} \geq 0. \tag{7.6.10}$$

(We will see in Section **??** that this implies exponential stability.) The quadratic constraint still holds if using a time-varying multiplier. Let $P : [0, \infty) \to \boldsymbol{\Pi}$ be (measurable and essentially) bounded. Note that $P(t)$ admits exactly the same structure as the constant multipliers above. For any such time-varying scaling we infer

$$\begin{pmatrix} \Delta(t) \\ I \end{pmatrix}^T P(t) \begin{pmatrix} \Delta(t) \\ I \end{pmatrix} \geq 0 \ \text{ for all } \ t \geq 0.$$

With the quadratically continuous mapping (why?)

$$\Sigma(x) := \int_0^\infty x(t)^T P(t) x(t) \, dt$$

on $\mathscr{L}_2^{k+l}$, we infer (**??**) by linearity. Hence, if there exists an $\varepsilon > 0$ with

$$\int_0^\infty \begin{pmatrix} w(t) \\ M(w)(t) \end{pmatrix}^T P(t) \begin{pmatrix} w(t) \\ M(w)(t) \end{pmatrix} dt \leq -\varepsilon \|w\|_2^2, \tag{7.6.11}$$

we can apply the more abstract Theorem **??** to infer that $\mathscr{I}_M(\Delta)^{-1}$ exists and has a uniformly bounded $\mathscr{L}_2$-gain. Again, (7.6.11) amounts to a scaled small-gain condition with time-varying scalings. If M can be described by

$$\dot{x} = A(t)x + B(t)w, \ \ z = C(t)x + D(t)w, \ \ x(0) = 0,$$

where $\dot{x} = A(t)x$ is exponentially stable, the validity of (7.6.11) can be characterized by a differential linear matrix inequality.

7.7   Notes and remarks

We stress that it is somewhat restrictive to consider a system as a mapping of signals, thus fixing what is considered as the system's input or output. It is not too difficult to extend our results to the more general and elegant behavioral approach [**?**, **?**], in line with the older literature in which systems have been defined as relations rather than mappings [**?**, **?**, **?**, **?**]. For reasons of clarity we have opted to confine ourselves to a few concepts in the pretty specific $\mathscr{L}_2$-setting which sets the stage for various modifications or extensions that have been suggested in the literature. Let us hint at a few of these generalizations in various respects. All notions are as easily introduced for $\mathscr{L}_p$- and $\mathscr{L}_{pe}$-spaces

with $1 \leq p \leq \infty$. The time-axis can be chosen as all nonnegative integers to investigate discrete-time systems. A mixture of continuous- and discrete-time allows to consider hybrid systems or systems with jumps. Moreover, the value set of the signals can be taken to be an arbitrary normed space, thus including infinite dimensional systems. Finally, the presented stability concepts are important samples out of a multitude of other possibilities. For example, stability of S if often defined to just require S to map $\mathscr{L}_2$ into $\mathscr{L}_2$ without necessarily having finite $\mathscr{L}_2$-gain. On the other hand in various applications it is important to qualify in a more refined fashion how $\|S(u)_T\|$ is related to $\|u_T\|$ for all T on the time set. For example, one could work with $\|S(u)_T\| \leq \gamma(\|u_T\|)$ to hold for all u, $T > 0$ and for some function $\gamma : [0,\infty) \to [0,\infty)$ in a specific class. We have confined ourselves to *affine* functions γ, but one could as well take the class of *monotonic* functions or include specific tangency and growth conditions for $T \to 0$ and $T \to \infty$ respectively, some of which can be found e.g. in [**?**, **?**]. Many of the abstract results to be presented in subsequent chapters could be formulated with general designer chosen stability properties that only need to obey certain (technical) axiomatic hypotheses.

The literature on the subject developed in this chapter is vast. We developed the subject on the basis of [**?**] and [**?**] but with modified proofs that are meant to offer openings for suitable generalizations. Theorems 7.35 and 7.40 are examples of such extension that go beyond those available in the literature and could be obtained relatively easily. In particular we believe it to be highly relevant to develop generic non-global IQC stability analysis tests (in order to limit the domains of definitions of the operators or the external signals) that are LMIable.

Zames-Falb multipliers have attracted substantial attention in the literature [**?**]. The tight parametrization of all multipliers in Section **??** goes beyond [**?**] in that it closes a gap for odd-monotone nonlinearities by making use of non-trivial results from approximation theory and matrix sum-of-squares relaxations. Notable extensions to repeated nonlinearities with example applications can be found in [**?**, **?**, **?**, **?**, **?**].

For results in order to optimize over both multiplier zeros and poles we refer to [**?**].

7.8 Exercises

Exercise 1
Consider the systems $R : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^l$, $S : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^l$ and $T : \mathscr{L}_{2e}^l \to \mathscr{L}_{2e}^m$.

(a) If R, S, T are bounded, show that $\|\alpha S\| = |\alpha| \|S\|$ ($\alpha \in \mathbb{R}$), $\|R + S\| \leq \|R\| + \|S\|$ and $\|TS\| \leq \|T\| \|S\|$.

(b) Show that the properties in a. hold for incrementally bounded systems and the incremental gain.

(c) If S is linear show that $\|S\|$ equals the infimal γ with

$$\|S(u)_T\| \leq \gamma \|u_T\| \quad \text{for all} \quad T > 0, \ u \in \mathscr{L}_{2e}^k.$$

Exercise 2

Suppose $S: \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^l$ is causal and has finite incremental gain. Moreover, assume that the restriction $S: \mathscr{L}_2^k \to \mathscr{L}_2^k$ has an inverse whose incremental gain is finite. Show that $S: \mathscr{L}_{2e} \to \mathscr{L}_{2e}$ itself has an inverse with finite incremental gain.

Exercise 3

Suppose w and z are two vectors in $\mathbb{R}^n$. Prove that

(a) There exists a $\Delta \in \mathbb{R}^{n \times n}$ with $\|\Delta\| \leq 1$ and $w = \Delta z$ iff $w^T w \leq z^T z$.

(b) There exists a $\delta \in \mathbb{R}$ with $|\delta| \leq 1$ and $w = \delta z$ iff $ww^T \leq zz^T$.

Exercise 4

For given $\Delta_j \in \mathbb{R}^{k \times l}$, define the set $\boldsymbol{\Delta} := \operatorname{conv}\{\Delta_1, \ldots, \Delta_N\}$. With fixed $Q = Q^T$, S, $R = R^T$ consider the function

$$f(\Delta) := \begin{pmatrix} \Delta \\ I \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} \Delta \\ I \end{pmatrix}.$$

(a) Prove that $Q \leq 0$ implies that $\Delta \to f(\Delta)$ is concave.

(b) Prove that if f is concave then

$$f(\Delta_j) > 0 \text{ for all } j = 1, \ldots, N \implies f(\Delta) > 0 \text{ for all } \Delta \in \boldsymbol{\Delta}. \qquad (7.8.1)$$

(c) Find weaker conditions on Q that lead to the the same implication (7.8.1).

Exercise 5 (MIMO Circle criterion)

Consider the system

$$\dot{x} = Ax + B\Delta(t, Cx)$$

where $\Delta: \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^k$ is any continuously differentiable function that satisfies, for two matrices K, L, the multi-variable sector condition

$$[\Delta(t,z) - Kz]^T [\Delta(t,z) - Lz] \leq 0 \text{ for all } (t,z) \in \mathbb{R} \times \mathbb{R}^l. \qquad (7.8.2)$$

Note that this is nothing but a static QC. Find a multiplier and the corresponding LMI that proves exponential stability. With $M(s) = C(sI - A)^{-1}B$ define $G(s) = (I - LM(s))(I - KM(s))^{-1}$. Show that the LMI you found has a solution iff

$$A + BKC \text{ is stable and } G(i\omega)^* + G(i\omega) > 0 \text{ for all } \omega \in \mathbb{R}.$$

(Hence G is strictly positive real. Note that this terminology is often used in the literature for a different property!) Is stability of A required for your arguments?

Exercise 6 (Popov criterion)

Consider the system

$$\dot{x} = Ax + B\Delta(Cx) \qquad (7.8.3)$$

with A Hurwitz and Δ a continuously differentiable nonlinearity $\Delta : \mathbb{R} \to \mathbb{R}$ that satisfies

$$0 \le z\Delta(z) \le z^2 \quad \text{for all } z \in \mathbb{R}.$$

Prove the following statements:

(a) Δ satisfies the static quadratic constraints

$$\tau \begin{pmatrix} \Delta(z) \\ z \end{pmatrix}^T \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta(z) \\ z \end{pmatrix} \ge 0$$

for all $z \in \mathbb{R}$ and $\tau \ge 0$.

(b) For any $z \in \mathbb{R}$

$$\int_0^z \Delta(\zeta)\,d\zeta \ge 0, \quad \int_0^z \zeta - \Delta(\zeta)\,d\zeta \ge 0.$$

If $z : [0, \infty) \to \mathbb{R}$ is continuously differentiable, then

$$\tau_1 \int_0^T z(t)\dot{z}(t) - \Delta(z(t))\dot{z}(t)\,dt \ge -\tau_1 \frac{1}{2}z(0)^2, \quad \tau_2 \int_0^T \Delta(z(t))\dot{z}(t)\,dt \ge -\tau_2 \frac{1}{2}z(0)^2$$

for $\tau_1, \tau_2 \ge 0$. (Substitution rule!)

(c) Suppose there exist X and $\tau, \tau_1, \tau_2 \ge 0$ such that

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^T \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & 0 \end{pmatrix}^T \begin{pmatrix} -2\tau & \tau \\ \tau & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ C & 0 \end{pmatrix} +$$

$$+ \begin{pmatrix} C & 0 \\ CA & CB \end{pmatrix}^T \begin{pmatrix} 0 & \tau_1 \\ \tau_1 & 0 \end{pmatrix} \begin{pmatrix} C & 0 \\ CA & CB \end{pmatrix} + \begin{pmatrix} 0 & I \\ CA & CB \end{pmatrix}^T \begin{pmatrix} 0 & -\tau_1 \\ \tau_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ CA & CB \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & I \\ CA & CB \end{pmatrix}^T \begin{pmatrix} 0 & \tau_2 \\ \tau_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ CA & CB \end{pmatrix} < 0.$$

Then the equilibrium $x_0 = 0$ of (7.8.3) is globally asymptotically Lyapunov stable. What can you say about exponential stability? Hint: Use a simple hard IQC argument. Note that, along trajectories of (7.8.3), one has $z = Cx$, $\dot{z} = CAx + CBw$ and $w = \Delta(z)$.

(d) Show that the condition in the previous exercise is equivalent to the existence of a symmetric K, $\nu \in \mathbb{R}$, $\tau > 0$ with

$$\begin{pmatrix} A^T K + KA & KB \\ B^T K & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ CA & CB \end{pmatrix}^T \begin{pmatrix} 0 & \nu \\ \nu & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ CA & CB \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & I \\ C & 0 \end{pmatrix}^T \begin{pmatrix} -2\tau & \tau \\ \tau & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ C & 0 \end{pmatrix} < 0.$$

(e) With $G(s) = C(sI - A)^{-1}B$, the LMI in the last exercise is solvable iff there exists a $q \in \mathbb{R}$ with

$$\mathrm{Re}((1 + qi\omega)G(i\omega)) < 1 \quad \text{for all } \omega \in \mathbb{R} \cup \{\infty\}.$$

This reveals the relation to the classical Popov criterion. Note that q is often assumed to be nonnegative what is, actually, a redundant hypothesis. Show with an example that the extra constraint $q \geq 0$ (or $v \geq 0$ in the LMI) introduces conservatism. (Think of a smart test using LMI-Lab to find an example.)

(f) Find an LMI condition for global asymptotic stability of

$$\dot{x} = Ax + \sum_{j=1}^{k} B_j \Delta_j(C_j x)$$

where the continuously differentiable $\Delta_j : \mathbb{R} \to \mathbb{R}$ satisfy the sector conditions $\beta_j z^2 \leq z \Delta_j(z) \leq \alpha_j z^2$ for all $z \in \mathbb{R}$.

Chapter 8

Robust controller synthesis

8.1 Robust controller design

So far we have presented techniques to design controllers for nominal stability and nominal performance. Previous chapters have been devoted to a thorough discussion of how to analyze, for a fixed stabilizing controller, robust stability or robust performance. For time-invariant or time-varying parametric uncertainties, we have seen direct tests formulated as searching for constant or parameter-dependent quadratic Lyapunov functions. For much larger classes of uncertainties, we have derived tests in terms of integral quadratic constraints (IQC's) that involve additional variables which have been called scalings or multipliers.

Typically, only those IQC tests with a class of multipliers that admit a state-space description as discussed in Sections **??**-**??** of Chapter 4 are amenable to a systematic output-feedback controller design procedure which is a reminiscent of the D/K-iteration in μ-theory. This will be the first subject of this chapter.

In a second section we consider as a particular information structure the robust state-feedback design problem. We will reveal that the search for static state-feedback gains which achieve robust performance can be transformed into a convex optimization problem.

The discussion is confined to the quadratic performance problem since most results can be extended in a pretty straightforward fashion to the other specifications considered in these notes.

8.1.1 Robust output-feedback controller design

If characterizing robust performance by an IQC, the goal in robust design is to find a controller *and* a multiplier such that, for the closed-loop system, the corresponding IQC test is satisfied. Hence, the multiplier appears as an extra unknown what makes the problem hard if not impossible to solve.

However, if the multiplier is held fixed, searching for a controller amounts to a nominal design problem that can be approached with the techniques described earlier. If the controller is held fixed, the analysis techniques presented in Chapter **??** can be used to find a suitable multiplier. Hence, instead of trying to search for a controller and a multiplier commonly, one iterates between the search for a controller with fixed multiplier and the search for a multiplier with fixed controller. This procedure is known from μ-theory as scalings/controller iteration or D/K iteration.

To be more concrete, we consider the specific *example* of achieving robust quadratic performance against time-varying parametric uncertainties as discussed in Section **??**.

The uncontrolled unperturbed system is described by (4.2.1). We assume that $w_1 \to z_1$ is the uncertainty channel and the uncontrolled uncertain system is described by including

$$w_1(t) = \Delta(t)z_1(t)$$

where $\Delta(.)$ varies in the set of continuous curves satisfying

$$\Delta(t) \in \mathbf{\Delta}_c := \text{conv}\{\Delta_1, ..., \Delta_N\} \quad \text{for all} \ \ t \geq 0.$$

We assume (w.l.o.g.) that

$$0 \in \text{conv}\{\Delta_1, ..., \Delta_N\}.$$

The performance channel is assumed to be given by $w_2 \to z_2$, and the performance index

$$P_p = \left(\begin{array}{cc} Q_p & S_p \\ S_p^T & R_p \end{array} \right), \ \ R_p \geq 0 \quad \text{with the inverse} \quad \tilde{P}_p^{-1} = \left(\begin{array}{cc} \tilde{Q}_p & \tilde{S}_p \\ \tilde{S}_p^T & \tilde{R}_p \end{array} \right), \ \ \tilde{Q}_p \leq 0$$

is used to define the quadratic performance specification

$$\int_0^\infty \left(\begin{array}{c} w_2(t) \\ z_2(t) \end{array} \right)^T P_p \left(\begin{array}{c} w_2(t) \\ z_2(t) \end{array} \right) dt \leq -\varepsilon \|w_2\|_2^2.$$

The goal is to design a controller that achieves robust stability and robust quadratic performance. We can guarantee both properties by finding a controller, a Lyapunov matrix $\mathscr{X}$, and a multiplier

$$P = \left(\begin{array}{cc} Q & S \\ S^T & R \end{array} \right), \ \ Q < 0, \ \ \left(\begin{array}{c} \Delta_j \\ I \end{array} \right)^T \left(\begin{array}{cc} Q & S \\ S^T & R \end{array} \right) \left(\begin{array}{c} \Delta_j \\ I \end{array} \right) > 0 \ \ \text{for all} \ \ j = 1, \dots, N \quad (8.1.1)$$

that satisfy the inequalities

$$
\mathcal{X} > 0, \quad
\begin{pmatrix}
I & 0 & 0 \\
\mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}_1 & \mathcal{X}\mathcal{B}_2 \\
0 & I & 0 \\
\mathcal{C}_1 & \mathcal{D}_1 & \mathcal{D}_{12} \\
0 & 0 & I \\
\mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_2
\end{pmatrix}^T
\left(\begin{array}{cc|cc|cc}
0 & I & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & Q & S & 0 & 0 \\
0 & 0 & S^T & R & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & Q_p & S_p \\
0 & 0 & 0 & 0 & S_p^T & R_p
\end{array}\right)
\begin{pmatrix}
I & 0 & 0 \\
\mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}_1 & \mathcal{X}\mathcal{B}_2 \\
0 & I & 0 \\
\mathcal{C}_1 & \mathcal{D}_1 & \mathcal{D}_{12} \\
0 & 0 & I \\
\mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_2
\end{pmatrix} < 0.
$$

(Recall that the condition on the left-upper block of P can be relaxed in particular cases what could reduce the conservatism of the test.)

If we apply the controller parameter transformation of Chapter **??**, we arrive at the synthesis matrix inequalities

$$
\boldsymbol{X}(v) > 0, \quad
\begin{pmatrix}
* \\ * \\ * \\ * \\ * \\ *
\end{pmatrix}^T
\left(\begin{array}{cc|cc|cc}
0 & I & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & Q & S & 0 & 0 \\
0 & 0 & \tilde{S}^T & R & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & Q_p & S_p \\
0 & 0 & 0 & 0 & S_p^T & R_p
\end{array}\right)
\begin{pmatrix}
I & 0 & 0 \\
\boldsymbol{A}(v) & \boldsymbol{B}_1(v) & \boldsymbol{B}_2(v) \\
0 & I & 0 \\
\boldsymbol{C}_1(v) & \boldsymbol{D}_1(v) & \boldsymbol{D}_{12}(v) \\
0 & 0 & I \\
\boldsymbol{C}_2(v) & \boldsymbol{D}_{21}(v) & \boldsymbol{D}_2(v)
\end{pmatrix} < 0.
$$

Unfortunately, there is no obvious way how to render these synthesis inequalities convex in *all* variables v, Q, S, R.

This is the reason why we consider, instead, the problem with a scaled uncertainty

$$
w_1(t) = [r\Delta(t)]z_1(t), \quad \Delta(t) \in \boldsymbol{\Delta}_c \tag{8.1.2}
$$

where the scaling factor is contained in the interval $[0, 1]$. Due to

$$
\begin{pmatrix} r\Delta \\ I \end{pmatrix}^T
\begin{pmatrix} Q & rS \\ rS^T & r^2 R \end{pmatrix}
\begin{pmatrix} r\Delta \\ I \end{pmatrix}
= r^2
\begin{pmatrix} \Delta \\ I \end{pmatrix}^T
\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}
\begin{pmatrix} \Delta \\ I \end{pmatrix},
$$

we conclude that the corresponding analysis or synthesis are given by (8.1.1) and

$$
\mathcal{X} > 0, \quad
\begin{pmatrix}
I & 0 & 0 \\
\mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}_1 & \mathcal{X}\mathcal{B}_2 \\
0 & I & 0 \\
\mathcal{C}_1 & \mathcal{D}_1 & \mathcal{D}_{12} \\
0 & 0 & I \\
\mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_2
\end{pmatrix}^T
\left(\begin{array}{cc|cc|cc}
0 & I & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & Q & rS & 0 & 0 \\
0 & 0 & rS^T & r^2 R & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & Q_p & S_p \\
0 & 0 & 0 & 0 & S_p^T & R_p
\end{array}\right)
\begin{pmatrix}
I & 0 & 0 \\
\mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}_1 & \mathcal{X}\mathcal{B}_2 \\
0 & I & 0 \\
\mathcal{C}_1 & \mathcal{D}_1 & \mathcal{D}_{12} \\
0 & 0 & I \\
\mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_2
\end{pmatrix} < 0
$$

$$\tag{8.1.3}$$

or

$$
\boldsymbol{X}(v) > 0, \quad
\begin{pmatrix} * \\ * \\ * \\ * \\ * \\ * \end{pmatrix}^T
\left(\begin{array}{cc|cc|cc}
0 & I & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & Q & rS & 0 & 0 \\
0 & 0 & rS^T & r^2R & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & Q_p & S_p \\
0 & 0 & 0 & 0 & S_p^T & R_p
\end{array} \right)
\left(\begin{array}{ccc}
I & 0 & 0 \\
\boldsymbol{A}(v) & \boldsymbol{B}_1(v) & \boldsymbol{B}_2(v) \\ \hline
0 & I & 0 \\
\boldsymbol{C}_1(v) & \boldsymbol{D}_1(v) & \boldsymbol{D}_{12}(v) \\ \hline
0 & 0 & I \\
\boldsymbol{C}_2(v) & \boldsymbol{D}_{21}(v) & \boldsymbol{D}_2(v)
\end{array} \right) < 0.
$$

(8.1.4)

For $r = 0$, we hence have to solve the nominal quadratic performance synthesis inequalities. If they are not solvable, the robust quadratic performance synthesis problem is not solvable either and we can stop. If they are solvable, the idea is to try to increase, keeping the synthesis inequalities feasible, the parameter r from zero to one. Increasing r is achieved by alternatingly maximizing r over v satisfying (8.1.4) (for fixed P) and by varying $\mathscr{X}$ and P in (8.1.3) (for a fixed controller).

The maximization of r proceeds along the following steps:

Initialization. Perform a nominal quadratic performance design by solving (8.1.4) for $r = 0$. Proceed if these inequalities are feasible and compute a corresponding controller.

After this initial phase, the iteration is started. The $j - 1$-st step of the iteration leads to a controller, a Lyapunov matrix $\mathscr{X}$, and a multiplier P that satisfy the inequalities (8.1.1) and (8.1.3) for the parameter $r = r_{j-1}$. Then it proceeds as follows:

First step: Fix the controller and maximize r by varying the Lyapunov matrix $\mathscr{X}$ and the scaling such that such that (8.1.1) and (8.1.3) hold. The maximal radius is denoted as $\widehat{r}_j$ and it satisfies $r_{j-1} \le \widehat{r}_j$.

Second step: Fix the resulting scaling P and find the largest r by varying the variables v in (8.1.4). The obtained maximum r_j clearly satisfies $\widehat{r}_j \le r_j$.

The iteration defines a sequence of radii

$$
r_1 \le r_2 \le r_3 \le \cdots
$$

and a corresponding controller that guarantee robust stability and robust quadratic performance for all uncertainties (8.1.2) with radius $r = r_j$.

If we are in the lucky situation that there is an index for which $r_j \ge 1$, the corresponding controller is robustly performing for all uncertainties with values in $\boldsymbol{\Delta}_c$ as desired, and we are done. However, if $r_j < 1$ for all indices, we cannot guarantee robust performance for $r = 1$, but we still have a guarantee of robust performance for $r = r_j$!

Before entering a brief discussion of this procedure, let us include the following remarks on the start-up and on the computations. If the nominal performance synthesis problem has a solution, the LMI's (8.1.1)-(8.1.3) do have a solution $\mathscr{X}$ and P for the resulting controller and for some - possibly

small - $r > 0$; this just follows by continuity. Hence the iteration does not get stuck after the first step. Secondly, for a fixed r, the first step of the iteration amounts to solving an analysis problem, and finding a solution v of (8.1.4) can be converted to an LMI problem. Therefore, the maximization of r can be performed by bisection.

Even if the inequalities (8.1.1)-(8.1.4) are solvable for $r = 1$, it can happen the the limit of r_j is smaller than one. As a remedy, one could consider another parameter to maximize, or one could modify the iteration scheme that has been sketched above. For example, it is possible to take the fine structure of the involved functions into account and to suggest other variable combinations that render the resulting iteration steps convex. Unfortunately, one cannot give general recommendations for modifications which guarantee success.

Remark. It should be noted that the controller/multiplier iteration can be extended to all robust performance tests that are based on families of dynamic IQC's which are described by real rational multipliers. Technically, one just requires a parametrization of the multipliers such that the corresponding analysis test (for a fixed controller) and the controller synthesis (for a fixed multiplier) both reduce to solving standard LMI problems.

8.1.2   Robust state-feedback controller design

For the same set-up as in the previous section we consider the corresponding synthesis problem if the state of the underlying system is measurable. According to our discussion in Section 4.5, the resulting synthesis inequalities read as

$$Q < 0, \quad \begin{pmatrix} \Delta_j \\ I \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} \Delta_j \\ I \end{pmatrix} > 0 \quad \text{for all} \ \ j = 1, \ldots, N$$

and

$$Y > 0, \quad \begin{pmatrix} * \\ * \\ * \\ * \\ * \\ * \end{pmatrix}^T \left(\begin{array}{cc|cc|cc} 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S & 0 & 0 \\ 0 & 0 & S^T & R & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q_p & S_p \\ 0 & 0 & 0 & 0 & S_p^T & R_p \end{array} \right) \left(\begin{array}{ccc} I & 0 & 0 \\ AY + BM & B_1 & B_2 \\ \hline 0 & I & 0 \\ C_1Y + E_1M & D_1 & D_{12} \\ \hline 0 & 0 & I \\ C_2Y + E_1M & D_{21} & D_2 \end{array} \right) < 0$$

in the variables Y, M, Q, S, R.

In this form these inequalities are *not convex*. However, we can apply the Dualization Lemma (Section 4.4.1) to arrive at the equivalent inequalities

$$\tilde{R} > 0, \quad \begin{pmatrix} I \\ -\Delta_j^T \end{pmatrix}^T \begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{pmatrix} \begin{pmatrix} I \\ -\Delta_j^T \end{pmatrix} < 0 \quad \text{for all} \ \ j = 1, \ldots, N$$

and $Y > 0$,

$$* \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{Q} & \tilde{S} & 0 & 0 \\ 0 & 0 & \tilde{S}^T & \tilde{R} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \tilde{Q}_p & \tilde{S}_p \\ 0 & 0 & 0 & 0 & \tilde{S}_p^T & \tilde{R}_p \end{pmatrix} \begin{pmatrix} -(AY+BM)^T & -(C_1Y+E_1M)^T & -(C_2Y+E_2M)^T \\ I & 0 & 0 \\ \hline -B_1^T & -D_1^T & -D_{21}^T \\ 0 & I & 0 \\ \hline -B_2^T & -D_{12}^T & -D_2^T \\ 0 & 0 & I \end{pmatrix} > 0$$

in the variables Y, M, $\tilde{Q}$, $\tilde{S}$, $\tilde{R}$. It turns out that these dual inequalities are all *affine* in the unknowns. Testing feasibility hence amounts to solving a standard LMI problem. If the LMI's are feasible, a robust static state-feedback gain is given by $\mathscr{D} = MY^{-1}$. This is one of the very few lucky instances in the world of designing robust controllers!

8.1.3   Affine parameter dependence

Let us finally consider the system

$$\begin{pmatrix} \dot{x} \\ z \\ y \end{pmatrix} = \begin{pmatrix} A(\Delta(t)) & B_1(\Delta(t)) & B(\Delta(t)) \\ C_1(\Delta(t)) & D(\Delta(t)) & E(\Delta(t)) \\ C(\Delta(t)) & F(\Delta(t)) & 0 \end{pmatrix} \begin{pmatrix} x \\ w \\ u \end{pmatrix}, \ \ \Delta(t) \in \mathrm{conv}\{\Delta_1, ..., \Delta_N\}$$

where the describing matrices depend *affinely* on the time-varying parameters. If designing output-feedback controllers, there is no systematic alternative to pulling out the uncertainties and applying the scalings techniques as in Section 8.1.1.

For robust state-feedback design there is an alternative without scalings. One just needs to directly solve the system of LMI's

$$Y > 0, \ \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix}^T \begin{pmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^T & R_p \end{pmatrix} \begin{pmatrix} I & 0 \\ A(\Delta_j)Y+B(\Delta_j)M & B_1(\Delta_j) \\ \hline 0 & I \\ C_1(\Delta_j)Y+E(\Delta_j)M & D(\Delta_j) \end{pmatrix} < 0, \ j = 1, \dots, N$$

(8.1.5)

in the variables Y and M.

For the controller gain $D_c = MY^{-1}$ we obtain

$$Y > 0, \ \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix}^T \begin{pmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^T & R_p \end{pmatrix} \begin{pmatrix} I & 0 \\ (A(\Delta_j)+B(\Delta_j)D_c)Y & B_1(\Delta_j) \\ \hline 0 & I \\ (C_1(\Delta_j)+E(\Delta_j)D_c)Y & D(\Delta_j) \end{pmatrix} < 0, \ j = 1, \dots, N$$

A convexity argument leads to

$$
Y > 0, \quad \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix}^T \left(\begin{array}{cc|cc} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^T & R_p \end{array} \right) \left(\begin{array}{cc} I & 0 \\ (A(\Delta(t)) + B(\Delta(t))D_c)Y & B_1(\Delta(t)) \\ \hline 0 & I \\ (C_1(\Delta(t)) + E(\Delta(t))D_c)Y & D(\Delta(t)) \end{array} \right) < 0
$$

for all parameter curves $\Delta(t) \in \mathrm{conv}\{\Delta_1, ..., \Delta_N\}$, and we can perform a congruence transformation as in Section 4.5 to get

$$
\mathscr{X} > 0, \quad \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix}^T \left(\begin{array}{cc|cc} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^T & R_p \end{array} \right) \left(\begin{array}{cc} I & 0 \\ \mathscr{X}(A(\Delta(t)) + B(\Delta(t))D_c) & \mathscr{X}B_1(\Delta(t)) \\ \hline 0 & I \\ (C_1(\Delta(t)) + E(\Delta(t))D_c) & D(\Delta(t)) \end{array} \right) < 0.
$$

These two inequalities imply, in turn, robust exponential stability and robust quadratic performance for the controlled system as seen in Section **??**.

We have proved that it suffices to directly solve the LMI's (8.1.5) to compute a robust static state-feedback controller. Hence, if the system's parameter dependence is affine, we have found two equivalent sets of synthesis inequalities that differ in the number of the involved variables and in the sizes of the LMI's that are involved. In practice, the correct choice is dictated by whatever system can be solved faster, more efficiently, or numerically more reliably.

Remark. Here is the reason why it is possible to directly solve the robust performance problem by state-feedback without scalings, and why this technique does, unfortunately, not extend to output-feedback control: The linearizing controller parameter transformation for state-feedback problems *does not involve the matrices that describe the open-loop system*, whereas that for that for ouptut-feedback problems indeed depends on the matrices A, B, C of the open-loop system description.

Let us conclude this chapter by stressing, again, that these techniques find straightforward extensions to other performance specifications. As an exercise, the reader is asked to work out the details of the corresponding results for the robust H_2-synthesis problem by state- or output-feedback.

8.2 Exercises

Exercise 1
This is an exercise on robust control. To reduce the complexity of programming, we consider a non-dynamic system only.

Suppose you have given the algebraic uncertain system

$$
\left(\begin{array}{c}
z_1 \\
z_2 \\
z_3 \\
z_4 \\
\hline
z \\
y_1 \\
y_2
\end{array}\right)
=
\left(\begin{array}{cccc|ccc}
0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0.5 & 0 & 0.5 & 0 & 1 & 0 & 1 \\
2a & 0 & a & 0 & 1 & 0 & 0 \\
0 & -2a & 0 & -a & 1 & 0 & 0 \\
\hline
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\left(\begin{array}{c}
w_1 \\
w_2 \\
w_3 \\
w_4 \\
\hline
w \\
u_1 \\
u_2
\end{array}\right),
$$

with a time-varying uncertainty

$$
\left(\begin{array}{c}
w_1 \\
w_2 \\
w_3 \\
w_4
\end{array}\right)
=
\left(\begin{array}{cccc}
\delta_1(t) & & & 0 \\
& \delta_1(t) & & \\
& & \delta_2(t) & \\
0 & & & \delta_2(t)
\end{array}\right)
\left(\begin{array}{c}
z_1 \\
z_2 \\
z_3 \\
z_4
\end{array}\right), \quad |\delta_1(t)| \le 0.7, \quad |\delta_2(t)| \le 0.7.
$$

As the performance measure we choose the L_2-gain of the channel $w \to z$.

(a) For the uncontrolled system and for each $a \in [0,1]$, find the minimal robust L_2-gain level of the channel $w \to z$ by applying the robust performance analysis test in Chapter 3 with the following class of scalings $P = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}$:

- P is as in μ-theory: Q, S, R are block-diagonal, $Q < 0$, R is related to Q (how?), and S is skew-symmetric.
- P is general with $Q < 0$.
- P is general with $Q_1 < 0$, $Q_2 < 0$, where Q_j denote the blocks $Q(1:2,1:2)$ and $Q(3:4,3:4)$ in Matlab notation.

Draw plots of the corresponding optimal values versus the parameter a and comment!

(b) For $a = 0.9$, apply the controller

$$
\left(\begin{array}{c} u_1 \\ u_2 \end{array}\right)
=
\left(\begin{array}{cc} 0 & 0 \\ 0 & k \end{array}\right)
\left(\begin{array}{c} y_1 \\ y_2 \end{array}\right)
$$

and perform the analysis test with the largest class of scalings for $k \in [-1,1]$. Plot the resulting optimal value over k and comment.

(c) Perform a controller/scaling iteration to minimize the optimal values for the controller structures

$$
\left(\begin{array}{c} u_1 \\ u_2 \end{array}\right)
=
\left(\begin{array}{cc} 0 & 0 \\ 0 & k_2 \end{array}\right)
\left(\begin{array}{c} y_1 \\ y_2 \end{array}\right)
\quad \text{and} \quad
\left(\begin{array}{c} u_1 \\ u_2 \end{array}\right)
=
\left(\begin{array}{cc} k_1 & k_{12} \\ k_{21} & k_2 \end{array}\right)
\left(\begin{array}{c} y_1 \\ y_2 \end{array}\right).
$$

Start from gain zero and plot the optimal values that can are reached in each step of the iteration to reveal how they decrease. Comment on the convergence.

(d) With the last full controller from the previous exercise for a performance level that is close to the limit, redo the analysis of the first part. Plot the curves and comment.

Chapter 9

Linear parameterically varying systems

Linear parameterically varying (LPV) systems are linear systems whose describing matrices depend on a time-varying parameter such that both the parameter itself and its rate of variation are known to be contained in pre-specified sets.

In robust control, the goal is to find *one fixed* controller that achieves robust stability and robust performance for all possible parameter variations, irrespective of which specific parameter curve does indeed perturb the system.

Instead, in LPV control, it is assumed that the parameter (and, possibly, its rate of variation), although not known a priori, is (are) on-line measurable. Hence the actual parameter value (and its derivative) can be used as extra information to control the system - the controller will turn out to depend on the parameter as well. We will actually choose also an LPV structure for the controller to be designed.

We would like to stress the decisive distinction to the control of time-varying systems: In the standard techniques to controlling time-varying systems, the model description is assumed to be known a priori over the whole time interval $[0, \infty)$. In LPV control, the model is assumed to be known, at time instant t, only over the interval $[0, t]$.

The techniques we would like to develop closely resemble those for robust control we have investigated earlier. It is possible to apply them

- to control certain classes of nonlinear systems

- to provide a systematic procedure for gain-scheduling with guarantees for stability and performance.

Before we explore these applications in more detail we would like to start presenting the available problem setups and solution techniques to LPV control.

9.1 General Parameter Dependence

Suppose that $\boldsymbol{\delta}_c, \dot{\boldsymbol{\delta}}_c \subset \mathbb{R}^m$ are two parameter sets such that

$$\boldsymbol{\delta}_c \times \dot{\boldsymbol{\delta}}_c \text{ is compact,}$$

and that the matrix valued function

$$\left(\begin{array}{c|cc} A(p) & B_p(p) & B(p) \\ \hline C_p(p) & D_p(p) & E(p) \\ C(p) & F(p) & 0 \end{array} \right) \text{ is continuous in } p \in \boldsymbol{\delta}_c. \tag{9.1.1}$$

Consider the *Linear Parameterically Varying (LPV) system* that is described as

$$\left(\begin{array}{c} \dot{x} \\ \hline z_p \\ y \end{array} \right) = \left(\begin{array}{c|cc} A(\delta(t)) & B_p(\delta(t)) & B(\delta(t)) \\ \hline C_p(\delta(t)) & D_p(\delta(t)) & E(\delta(t)) \\ C(\delta(t)) & F(\delta(t)) & 0 \end{array} \right) \left(\begin{array}{c} x \\ \hline w_p \\ u \end{array} \right), \ \ \delta(t) \in \boldsymbol{\delta}_c, \ \dot{\delta}(t) \in \dot{\boldsymbol{\delta}}_c. \tag{9.1.2}$$

We actually mean the family of systems that is obtained if letting $\delta(.)$ vary in the set of continuously differentiable parameter curves

$$\delta : [0, \infty) \to \mathbb{R}^m \text{ with } \delta(t) \in \boldsymbol{\delta}_c, \ \dot{\delta}(t) \in \dot{\boldsymbol{\delta}}_c \text{ for all } t \geq 0.$$

The signals admit the same interpretations as in Chapter 4: u is the control input, y is the measured output available for control, and $w_p \to z_p$ denotes the performance channel.

In *LPV control*, it is assumed that the parameter $\delta(t)$ is on-line measurable. Hence the actual value of $\delta(t)$ can be taken as extra information for the controller to achieve the desired design goal.

In view of the specific structure of the system description, we assume that the controller admits a similar structure. In fact, an *LPV controller* is defined by functions

$$\left(\begin{array}{cc} A_c(p) & B_c(p) \\ C_c(p) & D_c(p) \end{array} \right) \text{ that are continuous in } p \in \boldsymbol{\delta}_c \tag{9.1.3}$$

as

$$\left(\begin{array}{c} \dot{x}_c \\ u \end{array} \right) = \left(\begin{array}{cc} A_c(\delta(t)) & B_c(\delta(t)) \\ C_c(\delta(t)) & D_c(\delta(t)) \end{array} \right) \left(\begin{array}{c} x_c \\ y \end{array} \right)$$

with the following interpretation: It evolves according to linear dynamics that are defined at time-instant t via the actually measured value of $\delta(t)$.

Note that a robust controller would be simply defined with a constant matrix

$$\left(\begin{array}{cc} A_c & B_c \\ C_c & D_c \end{array} \right)$$

that does not depend on δ what clarifies the difference between robust controllers and LPV controllers.

The controlled system admits the description

$$\left(\begin{array}{c} \dot{\xi} \\ z_p \end{array} \right) = \left(\begin{array}{cc} \mathscr{A}(\delta(t)) & \mathscr{B}(\delta(t)) \\ \mathscr{C}(\delta(t)) & \mathscr{D}(\delta(t)) \end{array} \right) \left(\begin{array}{c} \xi \\ w_p \end{array} \right), \;\; \delta(t) \in \boldsymbol{\delta}_c, \;\; \dot{\delta}(t) \in \dot{\boldsymbol{\delta}}_c \qquad (9.1.4)$$

where the function

$$\left(\begin{array}{cc} \mathscr{A}(p) & \mathscr{B}(p) \\ \mathscr{C}(p) & \mathscr{D}(p) \end{array} \right) \;\; \text{is continuous in} \;\; p \in \boldsymbol{\delta}_c$$

and given as

$$\left(\begin{array}{cc|c} A(p)+B(p)D_c(p)C(p) & B(p)C_c(p) & B_p(p)+B(p)D_c(p)F(p) \\ B_c(p)C(p) & A_c(p) & B_c(p)F(p) \\ \hline C_p(p)+E(p)D_c(p)C(p) & E(p)C_c(p) & D_p(p)+E(p)D_c(p)F(p) \end{array} \right)$$

or

$$\left(\begin{array}{cc|c} A(p) & 0 & B_p(p) \\ 0 & 0 & 0 \\ \hline C_p(p) & 0 & D_p(p) \end{array} \right) + \left(\begin{array}{cc} 0 & B(p) \\ I & 0 \\ 0 & E(p) \end{array} \right) \left(\begin{array}{cc} A_c(p) & B_c(p) \\ C_c(p) & D_c(p) \end{array} \right) \left(\begin{array}{cc|c} 0 & I & 0 \\ C(p) & 0 & F(p) \end{array} \right).$$

To evaluate performance, we concentrate again on the quadratic specification

$$\int_0^\infty \left(\begin{array}{c} w(t) \\ z(t) \end{array} \right)^T P_p \left(\begin{array}{c} w(t) \\ z(t) \end{array} \right) dt \leq -\varepsilon \|w\|_2^2 \qquad (9.1.5)$$

with an index

$$P_p = \left(\begin{array}{cc} Q_p & S_p \\ S_p^T & R_p \end{array} \right), \;\; R_p \geq 0 \;\; \text{that has the inverse} \;\; \tilde{P}_p^{-1} = \left(\begin{array}{cc} \tilde{Q}_p & \tilde{S}_p \\ \tilde{S}_p^T & \tilde{R}_p \end{array} \right), \;\; \tilde{Q}_p \leq 0.$$

In order to abbreviate the formulation of the analysis result we introduce the following differential operator.

Definition 9.1 If $X : \boldsymbol{\delta}_c \ni p \to X(p) \in \mathbb{R}^{n \times n}$ is continuously differentiable, the continuous mapping

$$\partial X : \boldsymbol{\delta}_c \times \dot{\boldsymbol{\delta}}_c \to \mathbb{R}^{n \times n} \;\; \text{is defined as} \;\; \partial X(p,q) := \sum_{j=1}^m \frac{\partial X}{\partial p_j}(p) q_j.$$

Note that this definition is simply motivated by the fact that, along any continuously differentiable parameter curve $\delta(.)$, we have

$$\frac{d}{dt} X(\delta(t)) = \sum_{j=1}^m \frac{\partial X}{\partial p_j}(\delta(t)) \dot{\delta}_j(t) = \partial X(\delta(t), \dot{\delta}(t)). \qquad (9.1.6)$$

(We carefully wrote down the definitions and relations, and one should read all this correctly. X and ∂X are functions of the parameters $p \in \boldsymbol{\delta}_c$ and $q \in \dot{\boldsymbol{\delta}}_c$ respectively. In the definition of ∂X, no time-trajectories are involved. The definition of ∂X is just tailored to obtain the property (9.1.6) if plugging in a function of time.)

In view of the former discussion, the following analysis result comes as no surprise.

Theorem 9.2 *Suppose there exists a continuously differentiable $\mathscr{X}(p)$ defined for $p \in \boldsymbol{\delta}_c$ such that for all $p \in \boldsymbol{\delta}_c$ and $q \in \dot{\boldsymbol{\delta}}_c$ one has*

$$\mathscr{X}(p) > 0, \quad \left(\begin{array}{cc} \partial \mathscr{X}(p,q) + \mathscr{A}(p)^T \mathscr{X}(p) + \mathscr{X}(p)\mathscr{A}(p) & \mathscr{X}(p)\mathscr{B}(p) \\ \mathscr{B}(p)^T \mathscr{X}(p) & 0 \end{array} \right) +$$

$$+ \left(\begin{array}{cc} 0 & I \\ \mathscr{C}(p) & \mathscr{D}(p) \end{array} \right)^T P_p \left(\begin{array}{cc} 0 & I \\ \mathscr{C}(p) & \mathscr{D}(p) \end{array} \right) < 0. \quad (9.1.7)$$

Then there exists an $\varepsilon > 0$ such that, for each parameter curve with $\delta(t) \in \boldsymbol{\delta}_c$ and $\dot{\delta}(t) \in \dot{\boldsymbol{\delta}}_c$, the system (9.1.4) is exponentially stable and satisfies (9.1.5) if the initial condition is zero and if $w_p \in L_2$.

In view of our preparations the proof is a simple exercise that is left to the reader.

We can now use the same procedure as for LTI systems to arrive at the corresponding synthesis result. It is just required to obey that all the matrices are actually functions of $p \in \boldsymbol{\delta}_c$ or of $(p,q) \in \boldsymbol{\delta}_c \times \dot{\boldsymbol{\delta}}_c$. If partitioning

$$\mathscr{X} = \left(\begin{array}{cc} X & U \\ U^T & * \end{array} \right), \quad \mathscr{X}^{-1} = \left(\begin{array}{cc} Y & V \\ V^T & * \end{array} \right),$$

we can again assume w.l.o.g. that U, V have full row rank. (Note that this requires the compactness hypothesis on $\boldsymbol{\delta}_c$ and $\dot{\boldsymbol{\delta}}_c$. Why?) With

$$\mathscr{Y} = \left(\begin{array}{cc} Y & I \\ V^T & 0 \end{array} \right) \quad \text{and} \quad \mathscr{Z} = \left(\begin{array}{cc} I & 0 \\ X & U \end{array} \right)$$

we obtain the identities

$$\mathscr{Y}^T \mathscr{X} = \mathscr{Z} \quad \text{and} \quad I - XY = UV^T.$$

If we apply the differential operator ∂ to the first functional identity, we arrive at $(\partial \mathscr{Y})^T \mathscr{X} + \mathscr{Y}^T(\partial \mathscr{X}) = \partial \mathscr{Z}$. (Do the simple calculations. Note that ∂ is not the usual differentiation such that you cannot apply the standard product rule.) If we right-multiply $\mathscr{Y}$, this leads to

$$\mathscr{Y}^T(\partial \mathscr{X})\mathscr{Y} = (\partial \mathscr{Z})\mathscr{Y} - (\partial \mathscr{Y})^T \mathscr{Z}^T = \left(\begin{array}{cc} 0 & 0 \\ \partial X & \partial U \end{array} \right) \left(\begin{array}{cc} Y & I \\ V^T & 0 \end{array} \right) - \left(\begin{array}{cc} \partial Y & \partial V \\ 0 & 0 \end{array} \right) \left(\begin{array}{cc} I & X \\ 0 & U^T \end{array} \right)$$

and hence to

$$\mathscr{Y}^T(\partial \mathscr{X})\mathscr{Y} = \left(\begin{array}{cc} -\partial Y & -(\partial Y)X - (\partial V)U^T \\ (\partial X)Y + (\partial U)V^T & \partial X \end{array} \right).$$

If we introduce the transformed controller parameters

$$
\begin{pmatrix} K & L \\ M & N \end{pmatrix} = \begin{pmatrix} U & XB \\ 0 & I \end{pmatrix} \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} \begin{pmatrix} V^T & 0 \\ CY & I \end{pmatrix} + \begin{pmatrix} XAY & 0 \\ 0 & 0 \end{pmatrix} +
$$
$$
+ \begin{pmatrix} (\partial X)Y + (\partial U)V^T & 0 \\ 0 & 0 \end{pmatrix},
$$

a brief calculation reveals that

$$
\mathscr{Y}^T(\partial \mathscr{X} + \mathscr{A}^T \mathscr{X} + \mathscr{X} \mathscr{A})\mathscr{Y} = \begin{pmatrix} -\partial Y + \mathrm{sym}\,(AY + BM) & (A + BNC) + K^T \\ (A + BNC)^T + K & \partial X + \mathrm{sym}\,(AX + LC) \end{pmatrix}
$$

$$
\mathscr{Y}^T \mathscr{X} \mathscr{B} = \begin{pmatrix} B_p + BNF \\ XB_p + LF \end{pmatrix}, \quad \mathscr{C}\mathscr{Y} = \begin{pmatrix} C_pY + EM & C_p + ENC \end{pmatrix}, \quad \mathscr{D} = D_p + ENF
$$

where we used again the abbreviation $\mathrm{sym}\,(M) = M + M^T$. If compared to a parameter independent Lyapunov function, we have modified the transformation to K by $(\partial X)Y + (\partial U)V^T$ in order to eliminate this extra term that appears from the congruence transformation of $\partial \mathscr{X}$. If $\mathscr{X}$ is does not depend on p, $\partial \mathscr{X}$ vanishes identically and the original transformation is recovered.

We observe that L, M, N are functions of $p \in \boldsymbol{\delta}_c$ only, whereas K also depends on $q \in \dot{\boldsymbol{\delta}}_c$. In fact, this function has the structure

$$
K(p,q) = K_0(p) + \sum_{i=1}^m K_i(p)q_i \tag{9.1.8}
$$

(why?) and, hence, it is fully described by specifying

$$
K_i(p), \;\; i = 0, 1, \ldots, m
$$

that depend, as well, on $p \in \boldsymbol{\delta}_c$ only.

Literally as in Theorem 4.2 one can now prove the following synthesis result for LPV systems.

Theorem 9.3 *If there exists an LPV controller defined by* (9.1.3) *and a continuously differentiable* $\mathscr{X}(.)$ *defined for* $p \in \boldsymbol{\delta}_c$ *that satisfy* (9.1.7), *then there exist continuously differentiable functions* X, Y *and continuous functions* K_i, L, M, N *defined on* $\boldsymbol{\delta}_c$ *such that, with K given by* (9.1.8), *the inequalities*

$$
\begin{pmatrix} Y & I \\ I & X \end{pmatrix} > 0 \tag{9.1.9}
$$

and

$$
\left(\begin{array}{cc|c} -\partial Y + \mathrm{sym}\,(AY + BM) & (A + BNC) + K^T & B_p + BNF \\ (A + BNC)^T + K & \partial X + \mathrm{sym}\,(AX + LC) & XB_p + LF \\ \hline (B_p + BNF)^T & (XB_p + LF)^T & 0 \end{array} \right) +
$$
$$
+ \begin{pmatrix} * \\ * \end{pmatrix}^T P_p \left(\begin{array}{cc|c} 0 & 0 & I \\ C_pY + EM & C_p + ENC & D_p + ENF \end{array} \right) < 0 \tag{9.1.10}
$$

hold on $\boldsymbol{\delta}_c \times \dot{\boldsymbol{\delta}}_c$. Conversely, suppose the continuously differentiable X, Y and the continuous K_i, defining K as in (9.1.8), L, M, N satisfy these synthesis inequalities. Then one can factorize $I - XY = UV^T$ with continuously differentiable square and nonsingular U, V, and

$$\mathscr{X} = \begin{pmatrix} Y & V \\ I & 0 \end{pmatrix}^{-1} \begin{pmatrix} I & 0 \\ X & U \end{pmatrix} \tag{9.1.11}$$

$$\begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} = \begin{pmatrix} U & XB \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} K - XAY - [(\partial X)Y + (\partial U)V^T] & L \\ M & N \end{pmatrix} \begin{pmatrix} V^T & 0 \\ CY & I \end{pmatrix}^{-1} \tag{9.1.12}$$

render the analysis inequalities (9.1.7) satisfied.

Remark. Note that the formula (9.1.12) just emerges from the modified controller parameter transformation. We observe that the matrices B_c, C_c, D_c are functions of $p \in \boldsymbol{\delta}_c$ only. Due to the dependence of K on q and due to the extra term $U^{-1}[(\partial X)Y + (\partial U)V^T]V^{-T}$ in the formula for A_c, this latter matrix is a function that depends both on $p \in \boldsymbol{\delta}_c$ and $q \in \dot{\boldsymbol{\delta}}_c$. It has the same structure as K and can be written as

$$A_c(p,q) = A_0(p) + \sum_{i=1}^m A_i(p)q_i.$$

A straightforward calculation reveals that

$$A_i = U^{-1}[K_i V^{-T} - \frac{\partial X}{\partial p_i} Y V^{-T} - \frac{\partial U}{\partial p_i}], \quad i = 1, \dots, m.$$

Hence, to implement this controller, one indeed requires not only to measure $\delta(t)$ but also its rate of variation $\dot{\delta}(t)$. However, one could possibly exploit the freedom in choosing U and V to render $A_i = 0$ such that A_c does not depend on q any more. Recall that U and V need to be related by $I - XY = UV^T$; hence let us choose

$$V^T := U^{-1}(I - XY).$$

This leads to

$$A_i = U^{-1}[(K_i - \frac{\partial X}{\partial p_i} Y)(I - XY)^{-1}U - \frac{\partial U}{\partial p_i}], \quad i = 1, \dots, m.$$

Therefore, U should be chosen as a nonsingular solution of the system of first order partial differential equations

$$\frac{\partial U}{\partial p_i}(p) = [K_i(p) - \frac{\partial X}{\partial p_i}(p)Y(p)](I - X(p)Y(p))^{-1}U(p), \quad j = 1, \dots, m.$$

This leads to $A_i = 0$ such that the implementation of the LPV controller does not require any on-line measurements of the rate of the parameter variations. First order partial differential equations can be solved by the method of characteristics [19]. We cannot go into further details at this point.

In order to construct a controller that solves the LPV problem, one has to verify the solvability of the synthesis inequalities in the unknown *functions X, Y, K_i, L, M, N*, and for designing a controller, one has to find functions that solve them.

However, standard algorithms do not allow to solve functional inequalities directly. Hence we need to include a discussion of how to reduce these functional inequalities to finitely many LMI's in real variables.

First step. Since $q \in \dot{\boldsymbol{\delta}}_c$ enters the inequality (9.1.10) affinely, we can replace the set $\dot{\boldsymbol{\delta}}_c$, if convex, by its extreme points. Let us make the, in practice non-restrictive, assumption that this set has finitely many generators:

$$\dot{\boldsymbol{\delta}}_c = \mathrm{conv}\{\dot{\delta}^1, \ldots, \dot{\delta}^k\}.$$

Solving (9.1.9)-(9.1.10) over $(p,q) \in \boldsymbol{\delta}_c \times \dot{\boldsymbol{\delta}}_c$ is equivalent to solving (9.1.9)-(9.1.10) for

$$p \in \boldsymbol{\delta}_c, \;\; q \in \{\dot{\delta}^1, \ldots, \dot{\delta}^k\}. \tag{9.1.13}$$

Second step. Instead of searching over the set of all continuous functions, we restrict the search to a finite dimensional subspace thereof, as is standard in Ritz-Galerkin techniques. Let us hence choose basis functions

$$f_1(.), \ldots, f_l(.) \;\; \text{that are continuously differentiable on} \;\; \boldsymbol{\delta}_c.$$

Then all the functions to be found are assumed to belong to the subspace spanned by the functions f_j. This leads to the Ansatz

$$X(p) = \sum_{j=1}^{l} X_j f_j(p), \;\; Y(p) = \sum_{j=1}^{l} Y_j f_j(p)$$

$$K_i(p) = \sum_{j=1}^{l} K_j^i f_j(p), \;\; i = 0, 1, \ldots, m,$$

$$L(p) = \sum_{j=1}^{l} L_j f_j(p), \;\; M(p) = \sum_{j=1}^{l} M_j f_j(p), \;\; N(p) = \sum_{j=1}^{l} N_j f_j(p).$$

We observe

$$\partial X(p,q) = \sum_{j=1}^{l} X_j \partial f_j(p,q), \;\; \partial Y(p,q) = \sum_{j=1}^{l} Y_j \partial f_j(p,q).$$

If we plug these formulas into the inequalities (9.1.9)-(9.1.10), we observe that all the coefficient matrices enter affinely. After this substitution, (9.1.9)-(9.1.10) turns out to be a family of linear matrix inequalities in the

$$\text{matrix variables} \;\; X_j, Y_j, K_j^i, L_j, M_j, N_j$$

that is parameterized by (9.1.13). The variables of this system of LMI's are now real numbers; however, since the parameter p still varies in the infinite set $\boldsymbol{\delta}_c$, we have to solve infinitely many LMI's. This is, in fact, a so-called semi-infinite (not infinite dimensional as often claimed) convex optimization problem.

Third step. To reduce the semi-infinite system of LMI's to finitely many LMI's, the presently chosen route is to just fix a *finite subset*

$$\boldsymbol{\delta}_{\mathrm{finite}} \subset \boldsymbol{\delta}_c$$

and solve the LMI system in those points only. Hence the resulting family of LMI's is parameterized by

$$p \in \boldsymbol{\delta}_{\text{finite}} \text{ and } q \in \{\dot{\delta}^1, \ldots, \dot{\delta}^k\}.$$

We end up with a finite family of linear matrix inequalities in real valued unknowns that can be solved by standard algorithms. Since a systematic choice of points $\boldsymbol{\delta}_{\text{finite}}$ is obtained by gridding the parameter set, this last step is often called the gridding phase, and the whole procedure is said to be a gridding technique.

Remark on the second step. Due to Weierstraß' approximation theorem, one can choose a sequence of functions $f_1, f_2, \ldots$ on $\boldsymbol{\delta}_c$ such that the union of the subspaces

$$\mathscr{S}_\nu = \text{span}\{f_1, \ldots, f_\nu\}$$

is *dense* in the set of all continuously differentiable mappings on $\boldsymbol{\delta}_c$ with respect to the norm

$$\|f\| = \max\{|f(p)| \mid p \in \boldsymbol{\delta}_c\} + \sum_{j=1}^{m} \max\{|\frac{\partial f}{\partial p_j}(p)| \mid p \in \boldsymbol{\delta}_c\}.$$

This implies that, given any continuously differentiable g on $\boldsymbol{\delta}_c$ and any accuracy level $\varepsilon > 0$, one can find an index ν_0 such that there exists an $f \in \mathscr{S}_{\nu_0}$ for which

$$\forall p \in \boldsymbol{\delta}_c, \ q \in \dot{\boldsymbol{\delta}}_c : \ |g(p) - f(p)| \leq \varepsilon, \ |\partial g(p,q) - \partial f(p,q)| \leq \varepsilon.$$

(Provide the details.) Functions in the subspace $\mathscr{S}_\nu$ hence approximate any function g and its image ∂g under the differential operator ∂ up to arbitrary accuracy, if the index ν is chosen sufficiently large.

Therefore, if (9.1.9)-(9.1.10) viewed as functional inequalities do have a solution, then they have a solution if restricting the search over the finite dimensional subspace $\mathscr{S}_\nu$ for sufficiently large ν, i.e., if incorporating sufficiently many basis functions. However, the number of basis functions determines the number of variables in the resulting LMI problem. To keep the number of unknowns small requires an efficient choice of the basis functions what is, in theory and practice, a difficult problem for which one can hardly give any general recipes.

Remark on the third step. By compactness of $\boldsymbol{\delta}_c$ and continuity of all functions, solving the LMI's for $p \in \boldsymbol{\delta}_c$ or for $p \in \boldsymbol{\delta}_{\text{finite}}$ is equivalent if only the points are chosen sufficiently dense. A measure of density is the infimal ε such that the balls of radius ε around each of the finitely many points in $\boldsymbol{\delta}_{\text{finite}}$ already cover $\boldsymbol{\delta}_c$:

$$\boldsymbol{\delta}_c \subset \bigcup_{p_0 \in \boldsymbol{\delta}_{\text{finite}}} \{u \mid \|p - p_0\| \leq \varepsilon\}.$$

If the data functions describing the system are also differentiable in δ, one can apply the mean value theorem to provide explicit estimates of the accuracy of the required approximation. Again, however, it is important to observe that the number of LMI's to solve depends on the number of grid-points; hence one has to keep this number small in order to avoid large LMI's.

Remark on extensions. Only slight adaptations are required to treat all the other performance specifications (such as bounds on the L_2-gain and on the analogue of the H_2-norm or generalized H_2-norm

for time-varying systems) as well as the corresponding mixed problems as discussed in Chapter **??** in full generality. Note also that, for single-objective problems, the techniques to eliminate parameters literally apply; there is no need go into the details. In particular for solving gain-scheduling problems, it is important to observe that one can as well let the performance index depend on the measured parameter without any additional difficulty. As a designer, one can hence ask *different performance properties* in *different parameter ranges* what has considerable relevance in practical controller design.

Remark on robust LPV control. As another important extension we mention *robust LPV design*. It might happen that some parameters are indeed on-line measurable, whereas others have to be considered as unknown perturbations with which the controller cannot be scheduled. Again, it is straightforward to extend the robustness design techniques that have been presented in Chapter **??** from LTI systems and controllers to LPV systems and controllers. This even allows to include dynamic uncertainties if using IQC's to capture their properties. Note that the scalings that appear in such techniques constitute extra problem variables. In many circumstances it causes no extra technical difficulties to let these scalings also depend on the scheduling parameter what reduces the conservatism.

9.2 Polytopic Parameter Dependence

Suppose that the matrices (9.1.1) describing the system are *affine* functions on the set

$$\boldsymbol{\delta}_c = \text{conv}\{\delta^1, \ldots, \delta^k\}.$$

In that case we intend to search, as well, for an LPV controller that is defined with *affine* functions (9.1.3). Note that the describing matrices for the cosed-loop system are also *affine* in the parameter if

$$\begin{pmatrix} B \\ E \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} C & F \end{pmatrix} \quad \text{are parameter independent}$$

what is assumed from now on. Finally, we let $\mathscr{X}$ in Theorem 9.2 be *constant*.

Since $R_p \geq 0$, we infer that (9.1.7) is satisfied if and only if it holds for the generators $p = \delta^j$ of the set $\boldsymbol{\delta}_c$. Therefore, the analysis inequalities reduce to the finite set of LMI's

$$\mathscr{X} > 0, \quad \begin{pmatrix} \mathscr{A}(\delta^j)^T \mathscr{X} + \mathscr{X} \mathscr{A}(\delta^j) & \mathscr{X} \mathscr{B}(\delta^j) \\ \mathscr{B}(\delta^j)^T \mathscr{X} & 0 \end{pmatrix} +$$
$$+ \begin{pmatrix} 0 & I \\ \mathscr{C}(\delta^j) & \mathscr{D}(\delta^j) \end{pmatrix}^T P_p \begin{pmatrix} 0 & I \\ \mathscr{C}(\delta^j) & \mathscr{D}(\delta^j) \end{pmatrix} < 0 \quad \text{for all} \ \ j = 1, \ldots, k.$$

Under the present structural assumptions, the affine functions $\begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix}$ are transformed into

affine functions $\begin{pmatrix} K & L \\ M & N \end{pmatrix}$ under the controller parameter transformation as considered in the

previous section.

Then the synthesis inequalities (9.1.9)-(9.1.10) whose variables are the constant X and Y and the affine functions $\begin{pmatrix} K & L \\ M & N \end{pmatrix}$ turn out to be *affine* in the parameter p. This implies for the synthesis inequalities that we can replace the search over $\boldsymbol{\delta}_c$ without loss of generality by the search over the generators δ^j of this set. Therefore, solving the design problem amounts to testing whether the LMI's

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} > 0$$

and

$$\left(\begin{array}{cc|c} \mathrm{sym}\left(A(\delta^j)Y + BM(\delta^j)\right) & * & * \\ (A(\delta^j) + BN(\delta^j)C)^T + K(\delta^j) & \mathrm{sym}\left(A(\delta^j)X + L(\delta^j)C\right) & * \\ \hline (B_p(\delta^j) + BN(\delta^j)F)^T & (XB_p(\delta^j) + L(\delta^j)F)^T & 0 \end{array} \right) +$$
$$+ \begin{pmatrix} * \\ * \end{pmatrix}^T P_p \left(\begin{array}{cc|c} 0 & 0 & I \\ C_p(\delta^j)Y + EM(\delta^j) & C_p(\delta^j) + EN(\delta^j)C & D_p(\delta^j) + EN(\delta^j)F \end{array} \right) < 0$$

for $j = 1, \ldots, k$ admit a solution.

Since affine, the functions K, L, M, N are parameterized as

$$\begin{pmatrix} K(p) & L(p) \\ M(p) & N(p) \end{pmatrix} = \begin{pmatrix} K_0 & L_0 \\ M_0 & N_0 \end{pmatrix} + \sum_{i=1}^{m} \begin{pmatrix} K_i & L_i \\ M_i & N_i \end{pmatrix} p_i$$

with real matrices K_i, L_i, M_i, N_i. Hence, the synthesis inequalities form genuine linear matrix inequalities that can be solved by standard algorithms.

9.3 LFT System Description

Similarly as for our discussion of robust controller design, let us assume in this section that the LPV system is described as and LTI system

$$\begin{pmatrix} \dot{x} \\ \hline z_u \\ z_p \\ y \end{pmatrix} = \left(\begin{array}{c|ccc} A & B_u & B_p & B \\ \hline C_u & D_{uu} & D_{up} & E_u \\ C_p & D_{pu} & D_{pp} & E_p \\ C & F_u & F_p & 0 \end{array} \right) \begin{pmatrix} x \\ \hline w_u \\ w_p \\ u \end{pmatrix} \tag{9.3.1}$$

in wich the parameter enters via the uncertainty channel $w_u \rightarrow z_u$ as

$$w_u(t) = \Delta(t)z_u(t), \;\; \Delta(t) \in \boldsymbol{\Delta}_c. \tag{9.3.2}$$

The size and the structure of the possible parameter values $\Delta(t)$ is captured by the convex set

$$\boldsymbol{\Delta}_c := \mathrm{conv}\{\Delta_1, ..., \Delta_N\}$$
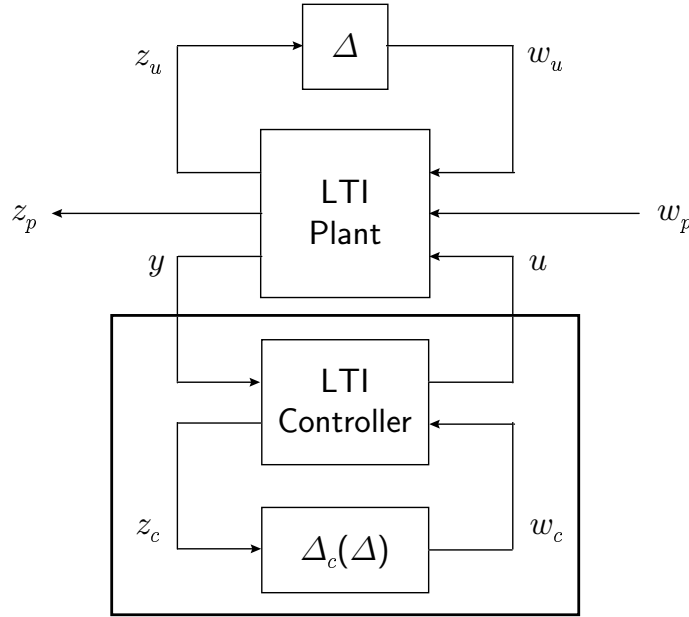
Figure 9.1: LPV system and LPV controller with LFT description

whose generators Δ_j are given explicitly. We assume w.l.o.g. that $0 \in \boldsymbol{\Delta}_c$. As before, we concentrate on the quadratic performance specification with index P_p imposed on the performance channel $w_p \to z_p$.

Adjusted to the structure of (9.3.1)-(9.3.2), we assume that the measured parameter curve enters the controller also in a linear fractional fashion. Therefore, we assume that the to-be-designed LPV controller is defined by scheduling the LTI system

$$\dot{x}_c = A_c x_c + B_c \begin{pmatrix} y \\ w_c \end{pmatrix}, \quad \begin{pmatrix} u \\ z_c \end{pmatrix} = C_c x_c + D_c \begin{pmatrix} y \\ w_c \end{pmatrix} \tag{9.3.3}$$

with the actual parameter curve entering as

$$w_c(t) = \Delta_c(\Delta(t))z_c(t). \tag{9.3.4}$$

The LPV controller is hence parameterized through the matrices A_c, B_c, C_c, D_c, and through a possibly non-linear matrix-valued scheduling function

$$\Delta_c(\Delta) \in \mathbb{R}^{n_r \times n_c} \quad \text{defined on} \ \boldsymbol{\Delta}_c.$$

Figure 9.1 illustrates this configuration.

The goal is to construct an LPV controller such that, for all admissible parameter curves, the controlled system is exponentially stable and, the quadratic performance specification with index P_p for the channel $w_p \to z_p$ is satisfied.
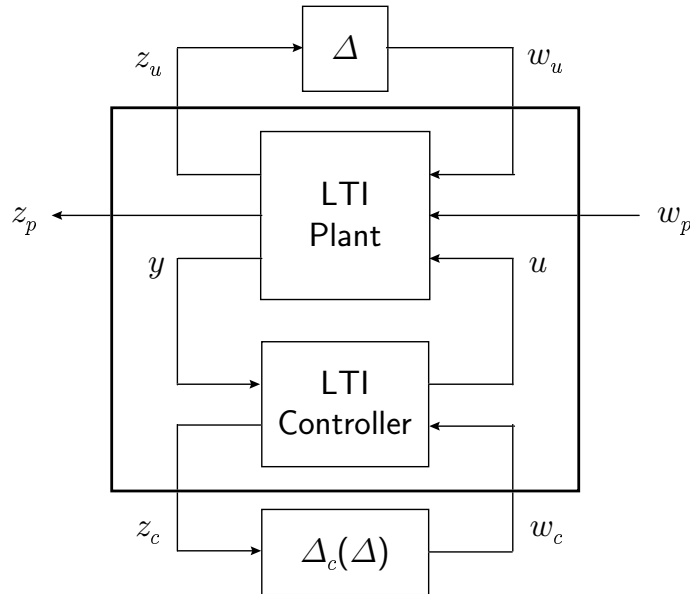
Figure 9.2: LPV system and LPV controller: alternative interpretation

The solution of this problem is approached with a simple trick. In fact, the controlled system can, alternatively, be obtained by scheduling the LTI system

$$
\left(\begin{array}{c} \dot{x} \\ \hline z_u \\ z_c \\ \hline z_p \\ y \\ w_c \end{array}\right) = \left(\begin{array}{c|cc|cc|c} A & B_u & 0 & B_p & B & 0 \\ \hline C_u & D_{uu} & 0 & D_{up} & E_u & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_c} \\ \hline C_p & D_{pu} & 0 & D_{uu} & E_p & 0 \\ C & F_u & 0 & F_p & 0 & 0 \\ 0 & 0 & I_{n_r} & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c} x \\ \hline w_u \\ w_c \\ \hline w_p \\ u \\ z_c \end{array}\right)
\tag{9.3.5}
$$

with the parameter as

$$
\left(\begin{array}{c} w_1 \\ w_c \end{array}\right) = \left(\begin{array}{cc} \Delta(t) & 0 \\ 0 & \Delta_c(\Delta(t)) \end{array}\right) \left(\begin{array}{c} z_1 \\ z_c \end{array}\right),
\tag{9.3.6}
$$

and then controlling this parameter dependent system with the LTI controller (9.3.3). Alternatively, we can interconnect the LTI system (9.3.5) with the LTI controller (9.3.3) to arrive at the LTI system

$$
\left(\begin{array}{c} \dot{x} \\ \hline z_u \\ z_c \\ z_p \end{array}\right) = \left(\begin{array}{c|ccc} \mathscr{A} & \mathscr{B}_u & \mathscr{B}_c & \mathscr{B}_p \\ \hline \mathscr{C}_u & \mathscr{D}_{uu} & \mathscr{D}_{uc} & \mathscr{D}_{up} \\ \mathscr{C}_c & \mathscr{D}_{cu} & \mathscr{D}_{cc} & \mathscr{D}_{cp} \\ \mathscr{C}_p & \mathscr{D}_{pu} & \mathscr{D}_{pc} & \mathscr{D}_{pp} \end{array}\right) \left(\begin{array}{c} x \\ \hline w_u \\ w_c \\ w_p \end{array}\right),
\tag{9.3.7}
$$

and then re-connect the parameter as (9.3.6). This latter interconnection order is illustrated in Figure 9.2.

Note that (9.3.5) is an extension of the original system (9.3.1) with an additional uncertainty channel $w_c \to z_c$ and with an additional control channel $z_c \to w_c$; the number n_r and n_c of the components of w_c and z_c dictate the size of the identity matrices I_{n_r} and I_{n_c} that are indicated by their respective indices.

Once the scheduling function $\Delta_c(\Delta)$ has been fixed, it turns out that (9.3.3) is a *robust controller* for the system (9.3.5) with uncertainty (9.3.6). The genuine robust control problem in which the parameter is not measured on-line would relate to the situation that $n_r = 0$ and $n_c = 0$ such that (9.3.5) and (9.3.1) are identical. In LPV control we have the extra freedom of being able to first extend the system as in (9.3.5) and design for this extended system a robust controller. It will turn out that this extra freedom will render the corresponding synthesis inequalities convex.

Before we embark on a solution of the LPV problem, let us include some further comments on the corresponding genuine robust control problem. We have seen in section 8.1.1 that the search for a robust controller leads to the problem of having to solve the matrix inequalities

$$
X(v) > 0, \quad
\begin{pmatrix} * \\ * \\ * \\ * \\ * \\ * \end{pmatrix}^T
\left(\begin{array}{cc|cc|cc}
0 & I & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & Q & S & 0 & 0 \\
0 & 0 & S^T & R & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & Q_p & S_p \\
0 & 0 & 0 & 0 & S_p^T & R_p
\end{array} \right)
\left(\begin{array}{ccc}
I & 0 & 0 \\
A(v) & B_u(v) & B_p(v) \\ \hline
0 & I & 0 \\
C_u(v) & D_{uu}(v) & D_{up}(v) \\ \hline
0 & 0 & I \\
C_p(v) & D_{pu}(v) & D_{pp}(v)
\end{array} \right) < 0
$$

$$
\begin{pmatrix} \Delta \\ I \end{pmatrix}^T
\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}
\begin{pmatrix} \Delta \\ I \end{pmatrix} > 0 \ \text{ for all } \ \Delta \in \Delta_c
$$

in the parameter v and in the multiplier $P = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}$.

Recall from our earlier discussion that one of the difficulties is a numerical tractable parameterization of the set of multipliers. This was the reason to introduce, at the expense of conservatism, the following subset of multipliers that admits a description in terms of finitely many LMI's:

$$
P := \left\{ P = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \mid Q < 0, \ \begin{pmatrix} \Delta_j \\ I \end{pmatrix}^T P \begin{pmatrix} \Delta_j \\ I \end{pmatrix} > 0 \ \text{ for } \ j = 1,\dots,N \right\}. \tag{9.3.8}
$$

Even after confining the search to v and $P \in P$, no technique is known how to solve the resulting still non-convex synthesis inequalities by standard algorithms.

In contrast to what we have seen for state-feedback design, the same is true of the dual inequalities

that read as

$$\boldsymbol{X}(v) > 0, \quad \begin{pmatrix} * \\ * \\ * \\ * \\ * \\ * \end{pmatrix}^T \left(\begin{array}{cc|cc|cc} 0 & X & 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{Q} & \tilde{S} & 0 & 0 \\ X & 0 & \tilde{S}^T & \tilde{R} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \tilde{Q}_p & \tilde{S}_p \\ 0 & 0 & 0 & 0 & \tilde{S}_p^T & \tilde{R}_p \end{array} \right) \left(\begin{array}{ccc} -\boldsymbol{A}(v)^T & -\boldsymbol{C}_u(v)^T & -\boldsymbol{C}_p(v)^T \\ I & 0 & 0 \\ -\boldsymbol{B}_u(v)^T & -\boldsymbol{D}_{uu}(v)^T & -\boldsymbol{D}_{pu}(v)^T \\ 0 & I & 0 \\ -\boldsymbol{B}_p(v)^T & -\boldsymbol{D}_{up}(v)^T & -\boldsymbol{D}_{pp}(v)^T \\ 0 & 0 & I \end{array} \right) > 0$$

$$\begin{pmatrix} I \\ -\Delta^T \end{pmatrix}^T \begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{pmatrix} \begin{pmatrix} I \\ -\Delta^T \end{pmatrix} < 0 \ \text{ for all } \ \Delta \in \boldsymbol{\Delta}_c.$$

Again, even confining the search to the set of multipliers

$$\tilde{\boldsymbol{P}} := \left\{ \tilde{P} = \begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{pmatrix} \mid \tilde{R} > 0, \ \begin{pmatrix} I \\ -\Delta_j^T \end{pmatrix}^T \tilde{P} \begin{pmatrix} I \\ -\Delta_j^T \end{pmatrix} < 0 \ \text{ for } \ j = 1, \ldots, N \right\} \quad (9.3.9)$$

does not lead to a convex feasibility problem.

Since non-convexity is caused by the multiplication of functions that depend on v with the multipliers, one could be lead to the idea that it might help to eliminate as many of the variables that are involved in v as possible. We can indeed apply the technique exposed in Section 4.4.4 and eliminate K, L, M, N.

For that purpose one needs to compute basis matrices

$$\Phi = \begin{pmatrix} \Phi^1 \\ \Phi^2 \\ \Phi^3 \end{pmatrix} \ \text{ of } \ \ker \begin{pmatrix} B^T & E_u^T & E_p^T \end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix} \Psi^1 \\ \Psi^2 \\ \Psi^3 \end{pmatrix} \ \text{ of } \ \ker \begin{pmatrix} C & F_u & F_p \end{pmatrix}$$

respectively. After elimination, the synthesis inequalities read as

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} > 0, \quad (9.3.10)$$

$$\Psi^T \left(\begin{array}{ccc} I & 0 & 0 \\ A & B_u & B_p \\ \hline 0 & I & 0 \\ C_u & D_{uu} & D_{up} \\ \hline 0 & 0 & I \\ C_p & D_{pu} & D_{pp} \end{array} \right)^T \left(\begin{array}{cc|cc|cc} 0 & X & 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S & 0 & 0 \\ 0 & 0 & S^T & R & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q_p & S_p \\ 0 & 0 & 0 & 0 & S_p^T & R_p \end{array} \right) \left(\begin{array}{ccc} I & 0 & 0 \\ A & B_u & B_p \\ \hline 0 & I & 0 \\ C_u & D_{uu} & D_{up} \\ \hline 0 & 0 & I \\ C_p & D_{pu} & D_{pp} \end{array} \right) \Psi < 0,$$

$$(9.3.11)$$

$$\Phi^T \left(\begin{array}{ccc} -A^T & -C_u^T & -C_p^T \\ I & 0 & 0 \\ -B_u^T & -D_{uu}^T & -D_{pu}^T \\ 0 & I & 0 \\ -B_p^T & -D_{pu}^T & -D_{pp}^T \\ 0 & 0 & I \end{array} \right)^T \left(\begin{array}{cc|cc|cc} 0 & X & 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{Q} & \tilde{S} & 0 & 0 \\ 0 & 0 & \tilde{S}^T & \tilde{R} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \tilde{Q}_p & \tilde{S}_p \\ 0 & 0 & 0 & 0 & \tilde{S}_p^T & \tilde{R}_p \end{array} \right) \left(\begin{array}{ccc} -A^T & -C_u^T & -C_p^T \\ I & 0 & 0 \\ -B_u^T & -D_{uu}^T & -D_{pu}^T \\ 0 & I & 0 \\ -B_p^T & -D_{pu}^T & -D_{pp}^T \\ 0 & 0 & I \end{array} \right) \Phi > 0$$

$$(9.3.12)$$

in the variables X, Y, and in the multiplier P and $\tilde{P}$ that are coupled as

$$\tilde{P} = \left(\begin{array}{cc} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{array} \right) = \left(\begin{array}{cc} Q & S \\ S^T & R \end{array} \right)^{-1} = P^{-1}. \tag{9.3.13}$$

Hence, after elimination, it turns out that the inequalities (9.3.10)-(9.3.12) are indeed affine in the unknowns X, Y, P and $\tilde{P}$. Unfortunately, non-convexity re-appears through the coupling (9.3.13) of the multipliers P and $\tilde{P}$.

Let us now turn back to the LPV problem where we allow, via the scheduling function $\Delta_c(\Delta)$ in the controller, extra freedom in the design process.

For guaranteeing stability and performance of the controlled system, we employ *extended* multipliers adjusted to the extended uncertainty structure (9.3.6) that are given as

$$P_e = \left(\begin{array}{c|c} Q_e & S_e \\ \hline S_e^T & R_e \end{array} \right) = \left(\begin{array}{cc|cc} Q & Q_{12} & S & S_{12} \\ Q_{21} & Q_{22} & S_{21} & S_{22} \\ \hline * & * & R & R_{12} \\ * & * & R_{21} & R_{22} \end{array} \right) \quad \text{with} \ \ Q_e < 0, \ R_e > 0 \tag{9.3.14}$$

and that satisfy

$$\left(\begin{array}{cc} \Delta & 0 \\ 0 & \Delta_c(\Delta) \\ \hline I & 0 \\ 0 & I \end{array} \right) P_e \left(\begin{array}{cc} \Delta & 0 \\ 0 & \Delta_c(\Delta) \\ \hline I & 0 \\ 0 & I \end{array} \right) > 0 \ \ \text{for all} \ \ \Delta \in \mathbf{\Delta}. \tag{9.3.15}$$

The corresponding dual multipliers $\tilde{P}_e = P_e^{-1}$ are partitioned similarly as

$$\tilde{P}_e = \left(\begin{array}{c|c} \tilde{Q}_e & \tilde{S}_e \\ \hline \tilde{S}_e^T & \tilde{R}_e \end{array} \right) = \left(\begin{array}{cc|cc} \tilde{Q} & \tilde{Q}_{12} & \tilde{S} & \tilde{S}_{12} \\ \tilde{Q}_{21} & \tilde{Q}_{22} & \tilde{S}_{21} & \tilde{S}_{22} \\ \hline * & * & \tilde{R} & \tilde{R}_{12} \\ * & * & \tilde{R}_{21} & \tilde{R}_{12} \end{array} \right) \quad \text{with} \ \ \tilde{Q}_e < 0, \ \tilde{R}_e > 0 \tag{9.3.16}$$

and they satisfy

$$\left(\begin{array}{cc} I & 0 \\ 0 & I \\ \hline -\Delta^T & 0 \\ 0 & -\Delta_c(\Delta)^T \end{array} \right)^T P_e \left(\begin{array}{cc} I & 0 \\ 0 & I \\ \hline -\Delta^T & 0 \\ 0 & -\Delta_c(\Delta)^T \end{array} \right) > 0 \ \ \text{for all} \ \ \Delta \in \mathbf{\Delta}.$$

As indicated by our notation, we observe that

$$\left(\begin{array}{cc} Q & S \\ S^T & R \end{array} \right) \in \boldsymbol{P} \quad \text{and} \quad \left(\begin{array}{cc} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{array} \right) \in \boldsymbol{\tilde{P}}$$

for the corresponding sub-matrices of P_e and $\tilde{P}_e$ respectively.

If we recall the description (9.3.6)-(9.3.7) of the controlled LPV system, the desired exponential stability and quadratic performance property is satisfied if we can find a Lyapunov matrix $\mathscr{X}$ and an extended scaling P_e with (9.3.14)-(9.3.15) such that

$$
\mathscr{X} > 0, \quad
\begin{pmatrix} * \\ * \\ \hline * \\ * \\ * \\ * \\ \hline * \\ * \end{pmatrix}^{T}
\left(\begin{array}{cc|cccc|cc}
0 & \mathscr{X} & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathscr{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & Q & Q_{12} & S & S_{12} & 0 & 0 \\
0 & 0 & Q_{21} & Q_{22} & S_{21} & S_{22} & 0 & 0 \\
0 & 0 & * & * & R & R_{12} & 0 & 0 \\
0 & 0 & * & * & R_{21} & R_{22} & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & Q_p & S_p \\
0 & 0 & 0 & 0 & 0 & 0 & S_p^{T} & R_p
\end{array}\right)
\left(\begin{array}{cccc}
I & 0 & 0 & 0 \\
\mathscr{A} & \mathscr{B}_u & \mathscr{B}_c & \mathscr{B}_p \\
\hline
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
\mathscr{C}_u & \mathscr{D}_{uu} & \mathscr{D}_{uc} & \mathscr{D}_{up} \\
\mathscr{C}_c & \mathscr{D}_{cu} & \mathscr{D}_{cc} & \mathscr{D}_{cp} \\
\hline
0 & 0 & 0 & I \\
\mathscr{C}_p & \mathscr{D}_{pu} & \mathscr{D}_{pc} & \mathscr{D}_{pp}
\end{array}\right) < 0.
$$
$$\tag{9.3.17}$$

We are now ready to formulate an LMI test for the existence of an LPV controller such that the controlled LPV system fulfills this latter analysis test.

Theorem 9.4 *The following statements are equivalent:*

(a) *There exists a controller* (9.3.3) *and a scheduling function* $\Delta_c(\Delta)$ *such that the controlled system as described by* (9.3.4)-(9.3.7) *admits a Lyapunov matrix* $\mathscr{X}$ *and a multiplier* (9.3.14)-(9.3.15) *that satisfy* (9.3.17).

(b) *There exist X, Y and multipliers* $P \in \boldsymbol{P}$, $\tilde{P} \in \tilde{\boldsymbol{P}}$ *that satisfy the linear matrix inequalities* (9.3.10)-(9.3.12).

Proof. Let us first prove $1 \Rightarrow 2$. We can apply the technique as described in Section 4.4.4 to eliminate the controller parameters in the inequality (9.3.17). According to Corollary 4.18, this leads to the coupling condition (4.4.23) and to the two synthesis inequalities (4.4.24)-(4.4.25). The whole point is to show that the latter two inequalities can indeed be simplified to (9.3.11)-(9.3.12). Let us illustrate this simplification for the first inequality only since a duality argument leads to the same conclusions for the second one.

With

$$
\Psi_e = \begin{pmatrix} \Psi^1 \\ \Psi^2 \\ 0 \\ \Psi^3 \end{pmatrix} \quad \text{as a basis matrix of } \ \ker \begin{pmatrix} C & F_u & 0 & F_p \\ 0 & 0 & I_{n_r} & 0 \end{pmatrix},
$$

the inequality that corresponds to (4.4.23) reads as

$$
\Psi_e^T \begin{pmatrix} * \\ * \\ * \\ * \\ * \\ * \\ * \\ * \end{pmatrix}^T
\left(\begin{array}{cc|cccc|cc}
0 & X & 0 & 0 & 0 & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & Q & Q_{12} & S & S_{12} & 0 & 0 \\
0 & 0 & Q_{21} & Q_{22} & S_{21} & S_{22} & 0 & 0 \\
0 & 0 & * & * & R & R_{12} & 0 & 0 \\
0 & 0 & * & * & R_{21} & R_{22} & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & Q_p & S_p \\
0 & 0 & 0 & 0 & 0 & 0 & S_p^T & R_p
\end{array} \right)
\left(\begin{array}{cccc}
I & 0 & 0 & 0 \\
A & B_u & 0 & B_p \\ \hline
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
C_u & D_{uu} & 0 & D_{up} \\
0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & I \\
C_p & D_{pu} & D_{pc} & D_{pp}
\end{array} \right) \Psi_e < 0.
$$

Due to the zero block in Ψ_e, it is obvious that this is the same as

$$
\Psi^T \begin{pmatrix} * \\ * \\ * \\ * \\ * \\ * \\ * \\ * \end{pmatrix}^T
\left(\begin{array}{cc|cccc|cc}
0 & X & 0 & 0 & 0 & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & Q & Q_{12} & S & S_{12} & 0 & 0 \\
0 & 0 & Q_{21} & Q_{22} & S_{21} & S_{22} & 0 & 0 \\
0 & 0 & * & * & R & R_{12} & 0 & 0 \\
0 & 0 & * & * & R_{21} & R_{22} & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & Q_p & S_p \\
0 & 0 & 0 & 0 & 0 & 0 & S_p^T & R_p
\end{array} \right)
\left(\begin{array}{ccc}
I & 0 & 0 \\
A & B_u & B_p \\ \hline
0 & I & 0 \\
0 & 0 & 0 \\
C_u & D_{uu} & D_{up} \\
0 & 0 & 0 \\ \hline
0 & 0 & I \\
C_p & D_{pu} & D_{pp}
\end{array} \right) \Psi < 0.
$$

The two zero block rows in the outer factors allow to simplify this latter inequality to (9.3.11), what finishes the proof of $1 \Rightarrow 2$.

The constructive proof of $2 \Rightarrow 1$ is more involved and proceeds in three steps. Let us assume that we have computed solutions X, Y and $P \in \mathbf{P}$, $\tilde{P} \in \tilde{\mathbf{P}}$ with (9.3.10)-(9.3.12).

First step: Extension of Scalings. Since $P \in \mathbf{P}$ and $\tilde{P} \in \tilde{\mathbf{P}}$, let us recall that we have

$$
\begin{pmatrix} \Delta \\ I \end{pmatrix}^T P \begin{pmatrix} \Delta \\ I \end{pmatrix} > 0 \;\; \text{and} \;\; \begin{pmatrix} I \\ -\Delta^T \end{pmatrix}^T \tilde{P} \begin{pmatrix} I \\ -\Delta^T \end{pmatrix} < 0 \;\; \text{for all} \;\; \Delta \in \mathbf{\Delta}. \tag{9.3.18}
$$

Due to $0 \in \mathbf{\Delta}_c$, we get $R > 0$ and $\tilde{Q} < 0$. Hence we conclude for the diagonal blocks of P that $Q < 0$ and $R > 0$, and for the diagonal blocks of $\tilde{P}$ that $\tilde{Q} > 0$ and $\tilde{R} < 0$. If we introduce

$$
Z = \begin{pmatrix} I \\ 0 \end{pmatrix} \;\; \text{and} \;\; \tilde{Z} = \begin{pmatrix} 0 \\ I \end{pmatrix}
$$

with the same row partition as P, these properties can be expressed as

$$
Z^T P Z < 0, \; \tilde{Z}^T P \tilde{Z} > 0 \;\; \text{and} \;\; Z^T \tilde{P} Z < 0, \; \tilde{Z}^T \tilde{P} \tilde{Z} > 0. \tag{9.3.19}
$$

If we observe that $\text{im}(\tilde{Z})$ is the orthogonal complement of $\text{im}(Z)$, we can apply the Dualization Lemma to infer

$$
\tilde{Z}^T P^{-1} \tilde{Z} > 0, \; Z^T P^{-1} Z < 0 \;\; \text{and} \;\; \tilde{Z}^T \tilde{P}^{-1} \tilde{Z} > 0, \; Z^T \tilde{P}^{-1} Z < 0. \tag{9.3.20}
$$

For the given P and $\tilde{P}$, we try to find an extension P_e with (9.3.14) such that the dual multiplier $\tilde{P}_e = P_e^{-1}$ is related to the given $\tilde{P}$ as in (9.3.16). After a suitable permutation, this amounts to finding an extension

$$\begin{pmatrix} P & T \\ T^T & T^T NT \end{pmatrix} \text{ with } \begin{pmatrix} \tilde{P} & * \\ * & * \end{pmatrix} = \begin{pmatrix} P & T \\ T^T & T^T NT \end{pmatrix}^{-1}, \tag{9.3.21}$$

where the specific parameterization of the new blocks in terms of a non-singular matrix T and some symmetric N will turn out convenient. Such an extension is very simple to obtain. However, we also need to obey the positivity/negativity constraints in (9.3.14) that amount to

$$\begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix}^T \begin{pmatrix} P & T \\ T^T & T^T NT \end{pmatrix} \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix} < 0 \tag{9.3.22}$$

and

$$\begin{pmatrix} \tilde{Z} & 0 \\ 0 & \tilde{Z} \end{pmatrix}^T \begin{pmatrix} P & T \\ T^T & T^T NT \end{pmatrix} \begin{pmatrix} \tilde{Z} & 0 \\ 0 & \tilde{Z} \end{pmatrix} > 0. \tag{9.3.23}$$

We can assume w.l.o.g. (perturb, if necessary) that $P - \tilde{P}^{-1}$ is non-singular. Then we set

$$N = (P - \tilde{P}^{-1})^{-1}$$

and observe that (9.3.21) holds for any non-singular T.

The main goal is to adjust T to render (9.3.22)-(9.3.23) satisfied. We will in fact construct the sub-blocks $T_1 = TZ$ and $T_2 = T\tilde{Z}$ of $T = (T_1 \ T_2)$. Due to (9.3.19), the conditions (9.3.22)-(9.3.23) read in terms of these blocks as (Schur)

$$T_1^T \left[N - Z(Z^T P Z)^{-1} Z^T \right] T_1 < 0 \text{ and } T_2^T \left[N - \tilde{Z}(\tilde{Z}^T P \tilde{Z})^{-1} \tilde{Z}^T \right] T_2 > 0. \tag{9.3.24}$$

If we denote by $n_+(S)$, $n_-(S)$ the number of positive, negative eigenvalues of the symmetric matrix S, we hence have to calculate $n_-(N - Z(Z^T P Z)^{-1} Z^T)$ and $n_+(N - \tilde{Z}(\tilde{Z}^T P \tilde{Z})^{-1} \tilde{Z}^T)$. Simple Schur complement arguments reveal that

$$n_- \begin{pmatrix} Z^T P Z & Z^T \\ Z & N \end{pmatrix} = n_-(Z^T P Z) + n_-(N - Z(Z^T P Z)^{-1} Z^T) =$$

$$= n_-(N) + n_-(Z^T P Z - Z^T N^{-1} Z) = n_-(N) + n_-(Z^T \tilde{P}^{-1} Z).$$

Since $Z^T P Z$ and $Z^T \tilde{P}^{-1} Z$ have the same size and are both negative definite by (9.3.19) and (9.3.20), we conclude $n_-(Z^T P Z) = n_-(Z^T \tilde{P}^{-1} Z)$. This leads to

$$n_-(N - Z(Z^T P Z)^{-1} Z^T) = n_-(N).$$

Literally the same arguments will reveal

$$n_+(N - \tilde{Z}(\tilde{Z}^T P \tilde{Z})^{-1} \tilde{Z}^T) = n_+(N).$$

These two relations imply that there exist T_1, T_2 with $n_-(N)$, $n_+(N)$ columns that satisfy (9.3.24). Hence the matrix $T = (T_1 \ T_2)$ has $n_+(N) + n_-(N)$ columns. Since the number of rows of T_1, T_2, Z,

$\tilde{Z}$, N are all identical, T is actually a *square* matrix. We can assume w.l.o.g. - by perturbing T_1 or T_2 if necessary - that the square matrix T is non-singular.

This finishes the construction of the extended multiplier (9.3.14). Let us observe that the dimensions of Q_{22}/R_{22} equal the number of columns of T_1/T_2 which are, in turn, given by the integers $n_-(N)/n_+(N)$.

Second Step: Construction of the scheduling function. Let us fix Δ and let us apply the Elimination Lemma to (9.3.15) with $\Delta_c(\Delta)$ viewed as the unknown. We observe that the solvability conditions of the Elimination Lemma just amount to the two inequalities (9.3.18). We conclude that for any $\Delta \in \mathbf{\Delta}$ one can indeed compute a $\Delta_c(\Delta)$ which satisfies (9.3.15).

Due to the structural simplicity, we can even provide an explicit formula which shows that $\Delta_c(\Delta)$ can be selected to depend smoothly on Δ. Indeed, by a straightforward Schur-complement argument, (9.3.15) is equivalent to

$$
\left(
\begin{array}{cc|cc}
U_{11} & U_{12} & (W_{11}+\Delta)^T & W_{21}^T \\
U_{21} & U_{22} & W_{12}^T & (W_{22}+\Delta_c(\Delta))^T \\
\hline
W_{11}+\Delta & W_{12} & V_{11} & V_{12} \\
W_{21} & W_{22}+\Delta_c(\Delta) & V_{21} & V_{22}
\end{array}
\right) > 0
$$

for $U = R_e - S_e^T Q_e^{-1} S_e > 0$, $V = -Q_e^{-1} > 0$, $W = Q_e^{-1} S_e$. Obviously this can be rewritten to

$$
\left(
\begin{array}{cc}
U_{22} & * \\
W_{22}+\Delta_c(\Delta) & V_{22}
\end{array}
\right)
-
\left(
\begin{array}{cc}
U_{21} & W_{12}^T \\
W_{21} & V_{21}
\end{array}
\right)
\left(
\begin{array}{cc}
U_{11} & (W_{11}+\Delta)^T \\
W_{11}+\Delta & V_{11}
\end{array}
\right)^{-1}
\left(
\begin{array}{cc}
U_{12} & W_{21}^T \\
W_{12} & V_{12}
\end{array}
\right) > 0
$$

in which $\Delta_c(\Delta)$ only appears in the off-diagonal position. Since we are sure that there does indeed exist a $\Delta_c(\Delta)$ that renders the inequality satisfied, the diagonal blocks must be positive definite. If we then choose $\Delta_c(\Delta)$ such that the off-diagonal block vanishes, we have found a solution of the inequality; this leads to the following explicit formula

$$
\Delta_c(\Delta) = -W_{22} +
\left(
\begin{array}{cc}
W_{21} & V_{21}
\end{array}
\right)
\left(
\begin{array}{cc}
U_{11} & * \\
W_{11}+\Delta & V_{11}
\end{array}
\right)^{-1}
\left(
\begin{array}{c}
U_{12} \\
W_{12}
\end{array}
\right)
$$

for the scheduling function. We note that $\Delta_c(\Delta)$ has the dimension $n_-(N) \times n_+(N)$.

Third Step: LTI controller construction. After having constructed the scalings, the last step is to construct an LTI controller and Lyapunov matrix that render the inequality (9.3.17) satisfied. We are confronted with a standard nominal quadratic design problem of which we are sure that it admits a solution, and for which the controller construction proceed along the steps that have been intensively discussed in Chapter **??**. ∎

We have shown that the LMI's that needed to be solved for designing an LPV controller are *identical* to those for designing a robust controller, with the only exception that the coupling condition (9.3.13) drops out. Therefore, the search for X and Y and for the multipliers $P \in \mathbf{P}$ and $\tilde{P} \in \tilde{\mathbf{P}}$ to satisfy (9.3.10)-(9.3.12) amounts to testing the feasibility of standard LMI's. Moreover, the controller

construction in the proof of Theorem 9.4 is constructive. Hence we conclude that we have found a full solution to the quadratic performance LPV control problem (including L_2-gain and dissipativity specifications) for full block scalings P_e that satisfy $Q_e < 0$. The more interesting general case without this still restrictive negativity hypotheses is dealt with in future work.

Remarks.

- The proof reveals that the scheduling function $\Delta_c(\Delta)$ has a many rows/colums as there are negative/positive eigenvalues of $P - \tilde{P}^{-1}$ (if assuming w.l.o.g. that the latter is non-singular.) If it happens that $P - \tilde{P}^{-1}$ is positive or negative definite, there is no need to schedule the controller at all; we obtain a controller that solves the robust quadratic performance problem.

- Previous approaches to the LPV problem [2, 15, 34, 51] were based on $\Delta_c(\Delta) = \Delta$ such that the controller is scheduled with an identical copy of the parameters. These results were based on block-diagonal parameter matrices and multipliers that were as well assumed block-diagonal. The use of full block scalings [46] require the extension to a more general scheduling function that is - as seen a posteriori - a quadratic function of the parameter Δ.

- It is possible to extend the procedure to H_2-control and to the other performance specifications in these notes. However, this requires restrictive hypotheses on the system description. The extension to general mixed problems seems nontrivial and is open in its full generality.

9.4 A Sketch of Possible Applications

It is obvious how to apply robust or LPV control techniques in linear design: If the underlying system is affected, possibly in a nonlinear fashion, by some possibly time-varying parameter (such as varying resonance poles and alike), one could strive

- either for designing a robust controller if the actual parameter changes are not available as on-line information

- or for constructing an LPV controller if the parameter (and its rate of variation) can be measured on-line.

As such the presented techniques can be a useful extension to the nominal design specifications that have been considered previously.

In a brief final and informal discussion we would like to point out possible applications of robust and LPV control techniques to the control of nonlinear systems:

- They clearly apply if one can systematically embed a nonlinear system in a class of linear systems that admit an LPV parameterization.

- Even if it is required to perform a heuristic linearization step, they can improve classical gain-scheduling design schemes for nonlinear systems since they lead to a one-shot construction of a family of linear controllers.

9.4.1   From Nonlinear Systems to LPV Systems

In order to apply the techniques discussed in these notes to nonlinear systems, one uses variations of what is often called *global linearization*.

Consider a nonlinear system described by

$$\dot{x} = f(x) \tag{9.4.1}$$

where we assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field.

If $f(0) = 0$, it is often possible to rewrite $f(x) = A(x)x$ with a smooth matrix valued mapping $A(.)$. If one can guarantee that the LPV system

$$\dot{x} = A(\delta(t))x$$

is exponentially stable, we can conclude that the nonlinear system

$$\dot{x} = A(x)x$$

has 0 as a globally exponentially stable equilibrium. Note that one can and should impose a priori bounds on the state-trajectories such as $x(t) \in M$ for some set M such that the stability of the LPV system only has to be assured for $\delta(t) \in M$; of course, one can then only conclude stability for trajectories of the nonlinear system that remain in M.

A slightly more general procedure allows to consider arbitrary system trajectories instead of equilibrium points (or constant trajectories) only. In fact, suppose $x_1(.)$ and $x_2(.)$ are two trajectories of (9.4.1). By the mean-value theorem, there exist

$$\eta_j(t) \in \operatorname{conv}\{x_1(t), x_2(t)\}$$

such that

$$\dot{x}_1(t) - \dot{x}_2(t) = f(x_1(t)) - f(x_2(t)) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(\eta_1(t)) \\ \vdots \\ \frac{\partial f_n}{\partial x}(\eta_n(t)) \end{pmatrix} (x_1(t) - x_2(t)).$$

Therefore, the increment $\xi(t) = x_1(t) - x_2(t)$ satisfies the LPV system

$$\dot{\xi}(t) = A(\eta_1(t), \ldots, \eta_n(t))\xi(t)$$

with parameters $\eta_1, \ldots, \eta_n$. Once this LPV system is shown to be exponentially stable, one can conclude that $\xi(t) = x_1(t) - x_2(t)$ converges exponentially to zero for $t \to \infty$. If $x_2(.)$ is a nominal

system trajectory (such as an equilibrium point or a given trajectory to be investigated), we can conclude that $x_1(t)$ approaches this nominal trajectory exponentially.

Finally, the following procedure is often referred to as *global linearization*. Let

$$\mathscr{F} \quad \text{be the closure of} \quad \text{conv}\{f_x(x) \mid x \in \mathbb{R}^n\}.$$

Clearly, $\mathscr{F}$ is a closed and convex subset of $\mathbb{R}^{n \times n}$. It is not difficult to see that any pair of trajectories $x_1(.), x_2(.)$ of (9.4.1) satisfies the linear differential inclusion

$$\dot{x}_1(t) - \dot{x}_2(t) \in \mathscr{F}(x_1(t) - x_2(t)). \tag{9.4.2}$$

Proof. Fix any t and consider the closed convex set

$$\mathscr{F}[x_1(t) - x_2(t)] \subset \mathbb{R}^n.$$

Suppose this set is contained in the negative half-space defined by the vector $y \in \mathbb{R}^n$:

$$y^T \mathscr{F}[x_1(t) - x_2(t)] \leq 0.$$

Due to the mean-value theorem, there exists a $\xi \in \text{conv}\{x_1(t), x_2(t)\}$ with

$$y^T[\dot{x}_1(t) - \dot{x}_2(t)] = y^T[f(x_1(t)) - f(x_2(t))] = y^T f_x(\xi)[x_1(t) - x_2(t)].$$

Since $f_x(\xi) \in \mathscr{F}$, we infer

$$y^T[\dot{x}_1(t) - \dot{x}_2(t)] \leq 0.$$

Hence $\dot{x}_1(t) - \dot{x}_2(t)$ is contained, as well, in the negative half-space defined by y. Since $\mathscr{F}$ is closed and convex, we can indeed infer (9.4.2) as desired. ∎

To analyze the stability of the differential inclusion, one can cover the set $\mathscr{F}$ by the convex hull of finitely many matrices A_j and apply the techniques that have been presented in these notes.

Remarks. Of course, there are many other possibilities to embed nonlinear systems in a family of linear systems that depend on a time-varying parameter. Since there is no general recipe to transform a given problem to the LPV scenario, we have only sketched a few ideas. Although we concentrated on stability analysis, these ideas straightforwardly extend to various nominal or robust performance design problems what is a considerable advantage over other techniques for nonlinear systems. This is particularly important since, in practical problems, non-linearities are often highly structured and not all states enter non-linearly. For example, in a stabilization problem, one might arrive at a system

$$\dot{x} = A(y)x + B(y)u, \quad y = Cx$$

where u is the control input and y is the measured output that captures, as well, those states that enter the system non-linearly. We can use the LPV techniques to design a stabilizing LPV controller for this system. Since y is the scheduling variable, this controller will depend, in general, non-linearly on y; hence LPV control amounts to a systematic technique to design nonlinear controllers for nonlinear systems 'whose non-linearities can be measured'.

9.4.2 Gain-Scheduling

A typical engineering technique to attack design problems for nonlinear systems proceeds as follows: Linearize the system around a couple of operating points, design good linear controllers for each of these points, and then glue these linear controllers together to control the nonlinear system.

Although this scheme seems to work reasonably well in many practical circumstances, there are considerable drawbacks:

- There is no general recipe how to glue controllers together. It is hard to discriminate between several conceivable controller interpolation techniques.

- It is not clear how to design the linear controllers such that, after interpolation, the overall controlled system shows the desired performance.

- There are no guarantees whatsoever that the overall system is even stabilized, not to speak of guarantees for performance. Only through nonlinear simulations one can roughly assess that the chosen design scenario has been successful.

Based on LPV techniques, one can provide a recipe to systematically design a family of linear controllers that is scheduled on the operating point without the need for ad-hoc interpolation strategies. Moreover, one can provide, at least for the linearized family of systems, guarantees for stability and performance, even if the system undergoes rapid changes of the operating condition.

Again, we just look at the stabilization problem and observe that the extensions to include as well performance specifications are straightforward.

Suppose a nonlinear system

$$\dot{x} = a(x,u), \;\; y = c(x,u) - r \tag{9.4.3}$$

has x as its state, u as its control, r as a reference input, and y as a tracking error output that is also the measured output. We assume that, for each reference input r, the system admits a unique equilibrium (operating condition)

$$0 = a(x_0(r), u_0(r)), \;\; 0 = c(x_0(r), u_0(r)) - r$$

such that $x_0(.)$, $u_0(.)$ are smooth in r. (In general, one applies the implicit function theorem to guarantee the existence of such a parameterized family of equilibria under certain conditions. In practice, the calculation of these operating points is the first step to be done.)

The next step is to linearize the the system around each operating point to obtain

$$\dot{x} = f_x(x_0(r), u_0(r))x + f_u(x_0(r), u_0(r))u, \;\; y = c_x(x_0(r), u_0(r))x + c_u(x_0(r), u_0(r))u - r.$$

This is indeed a family of linear systems that is parameterized by r.

In standard gain-scheduling, linear techniques are used to find, for each r, a good tracking controller for each of these systems, and the resulting controllers are then somehow interpolated.

At this point we can exploit the LPV techniques to systematically design an LPV controller that achieves good tracking for all reference trajectories in a certain class, even if these references vary quickly with time. This systematic approach directly leads to a family of linear systems, where the interpolation step is taken care of by the algorithm. Still, however, one has to confirm by nonlinear simulations that the resulting LPV controller works well for the original nonlinear system. Note that the Taylor linearization can sometimes be replaced by global linearization (as discussed in the previous section) what leads to a priori guarantees for the controlled nonlinear system.

Again, this was only a very brief sketch of ideas to apply LPV control in gain-scheduling, and we refer to [22] for a broader exposition of gain-scheduling in nonlinear control.

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