Sahand University of Technology
Control Engineering Department

LMI in Control Systems

Hadi Azmi

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Norms:

A norm is a way to measure the size of a vector, a matrix, a tensor, or a function. In this lecture we will learn the norm defination and properties. First of all we devided the norm lecture to four important subsections.

A) Norm of Vectors

B) Norm of Matrix

C) Norm of Signals

D) Norm of Systems

Vector Norms

A norm describes the size or 'length' of a vector $\mathbf{x}$. Although we are used to the traditional Euclidean norm $\sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_N|^2}$, a variety of norms can be defined, with the following conditions:

1. A norm $\|\mathbf{x}\|$ is always ≥ 0, with $\|\mathbf{x}\| = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$
2. $\|c\mathbf{x}\| = |c|\|\mathbf{x}\|$ for any scalar c
3. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality)

Note that there is an enormous number of different ways to measure the size of a vector, ie: different vector norms. In this course, we will pay particular attention to the set of norms called ℓ_p norms. For a given value of $p \geq 1$, the ℓ_p norm of a vector is defined as:

$$\|\mathbf{x}\|_p = \left[\sum_n |x_n|^p\right]^{1/p}$$

It can be shown that for any $p > 0$, $\|\cdot\|_p$ defines a vector norm. The following p-norms are of particular interest:

- $p = 1$: The ℓ_1-*norm*

$$\|\mathbf{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

- $p = 2$: The ℓ_2-*norm* or *Euclidean norm*

$$\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\mathbf{x}^T\mathbf{x}}$$

- $p = \infty$: The ℓ_∞-*norm*

$$\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

It can be shown that the ℓ_2-norm satisfies the *Cauchy-Bunyakovsky-Schwarz* inequality

$$|\mathbf{x}^T\mathbf{y}| \leq \|\mathbf{x}\|_2\|\mathbf{y}\|_2$$

for any vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$. This inequality is useful for showing that the ℓ_2-norm satisfies the triangle inequality. It is a special case of the *Holder* inequality

$$|\mathbf{x}^T\mathbf{y}| \leq \|\mathbf{x}\|_p\|\mathbf{y}\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

We say that two vector norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are *equivalent* if there exists constants C_1 and C_2, that are independent of $\mathbf{x}$, such that for any vector $\mathbf{x} \in \mathbb{R}^n$,

$$C_1\|\mathbf{x}\|_\alpha \leq \|\mathbf{x}\|_\beta \leq C_2\|\mathbf{x}\|_\alpha.$$

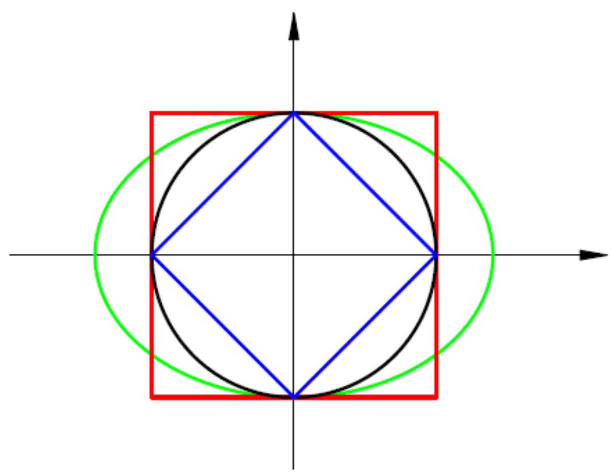
It follows that if two norms are equivalent, then a sequence of vectors that converges to a limit with respect to one norm will converge to the same limit in the other. It can be shown that all ℓ_p-norms are equivalent. In particular, if $\mathbf{x} \in \mathbb{R}^n$, then

$$\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq \sqrt{n}\|\mathbf{x}\|_2,$$

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_\infty,$$

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_1 \leq n\|\mathbf{x}\|_\infty.$$

***Example**: plot the norm veector* $\|x\| = 1$



$\|\mathbf{x}\| = (x_1^2 + x_2^2)^{1/2}$	black
$\|\mathbf{x}\| = \left(\frac{1}{2}x_1^2 + x_2^2\right)^{1/2}$	green
$\|\mathbf{x}\| = \|x_1\| + \|x_2\|$	blue
$\|\mathbf{x}\| = \max(\|x_1\|, \|x_2\|)$	red

Matrix Norms

It is also very useful to be able to measure the magnitude of a matrix, or the distance between matrices. However, it is not sufficient to simply define the norm of an $m \times n$ matrix A as the norm of an mn-vector $\mathbf{x}$ whose components are the entries of A. We instead define a *matrix norm* to be a function $\|\cdot\| : \mathbb{R}^{m\times n} \to \mathbb{R}$ that has the following properties:

- $\|A\| \geq 0$ for any $A \in \mathbb{R}^{m\times n}$, and $\|A\| = 0$ if and only if $A = 0$
- $\|\alpha A\| = |\alpha|\|A\|$ for any $m \times n$ matrix A and scalar α
- $\|A + B\| \leq \|A\| + \|B\|$ for any $m \times n$ matrices A and B

Another property that is often, but not always, included in the definition of a matrix norm is the *submultiplicative property*: if A is $m \times n$ and B is $n \times p$, we require that

$$\|AB\| \leq \|A\|\|B\|.$$

This is particularly useful when A and B are square matrices.

Any vector norm *induces* a matrix norm. It can be shown that given a vector norm, defined appropriately for m-vectors and n-vectors, the function $\|\cdot\| : \mathbb{R}^{m\times n} \to \mathbb{R}$ defined by

$$\|A\| = \sup_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

is a matrix norm. It is called the *natural*, or *induced*, matrix norm. Furthermore, if the vector norm is a ℓ_p-norm, then the induced matrix norm satisfies the submultiplicative property.

The following matrix norms are of particular interest:

- The ℓ_1-*norm*:

$$\|A\|_1 = \max_{\|\mathbf{x}\|_1=1} \|A\mathbf{x}\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}|.$$

  That is, the ℓ_1-norm of a matrix is its maximum column sum.

- The ℓ_∞-*norm*:

$$\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty=1} \|A\mathbf{x}\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|.$$

  That is, the ℓ_∞-norm of a matrix is its maximum *row* sum.

- The ℓ_2-*norm*:

$$\|A\|_2 = \max_{\|\mathbf{x}\|_2=1} \|A\mathbf{x}\|_2.$$

Theorem *Let A be an $m \times n$ matrix. Then*

$$\|A\|_\infty = \max_{1\leq i\leq m} \left[\sum_{j=1}^{n} |a_{ij}|\right] \qquad \text{(max absolute row sum)}$$

$$\|A\|_1 = \max_{1\leq j\leq n} \left[\sum_{i=1}^{m} |a_{ij}|\right] \qquad \text{(max absolute column sum)}$$

Example Determine $\|A\|_\infty$ and $\|A\|_1$ where

$$A = \begin{pmatrix} 1 & 2 & -4 \\ 3 & 0 & 12 \\ -20 & -1 & 2 \end{pmatrix}$$

We have

$$\|A\|_1 = \max\{(1+3+20), (2+1), (4+12+2)\} = \max\{24, 3, 18\} = 24$$

$$\|A\|_\infty = \max\{(1+2+4), (3+12), (20+1+2)\} = \max\{7, 15, 23\} = 23$$

Algebraic solution of Linear Least-Squares Problems

Let us consider our standard linear problem as defined before:

$$\mathbf{A}\mathbf{x} = \mathbf{y}$$

and focus specifically on the case where an exact solution does not exist, for instance because the measurement vector $\mathbf{y}$ has noise in it. In this case, a very common approach to solving this problem is to seek the best solution $\hat{\mathbf{x}}$ in the sense of matching the measurement vector as closely as possible. How do we define 'as closely as possible'? Here is where the norms come in, as a measure of size (or distance):

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_p$$

Using basic calculus, in order to minimize $\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2$ we can look for the choice of $\mathbf{x}$ (which we call $\hat{\mathbf{x}}$) such that the gradient of $\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2$ is zero, ie:

$$\mathbf{A}^T(\mathbf{A}\hat{\mathbf{x}} - \mathbf{y}) = 0$$

(let's stick to real-valued vectors and matrices, for simplicity), which can be rewritten as:

$$\mathbf{A}^T\mathbf{A}\hat{\mathbf{x}} = \mathbf{A}^T\mathbf{y}$$

Now, if the matrix $\mathbf{A}^T\mathbf{A}$ has an inverse matrix $(\mathbf{A}^T\mathbf{A})^{-1}$, we can apply it to both sides of the equation above, ie:

$$(\mathbf{A}^T\mathbf{A})^{-1}(\mathbf{A}^T\mathbf{A})\hat{\mathbf{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y}$$

or in other words,

$$\hat{\mathbf{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y}$$

And this is our linear least squares solution.

Signal Norms

We consider real valued signals[1] that are piecewise continuous functions of time $t \in [0, \infty)$. In this section we introduce some different norms for these signals.

Definition (Norm on a Vector Space) Let $\mathbf{V}$ be a vector space, a given non-negative function $\phi : \mathbf{V} \to \mathbf{R}^+$ is a norm on $\mathbf{V}$ if it satisfies

$$\phi(v) \geq 0, \quad \phi(v) = 0 \Leftrightarrow v = 0$$

$$\phi(\alpha v) = |\alpha| \phi(v)$$

$$\phi(v + w) \leq \phi(v) + \phi(w)$$

for all $\alpha \in \mathbf{R}$ and $v, w \in \mathbf{V}$.

A norm is defined on a vector space. To apply this concept to the case of signals, it is necessary to define sets of signals that are vector spaces. This is the case of the signal spaces described below.

$\mathbf{L}_1$-*Space and* $\mathbf{L}_1$-*Norm*

The $\mathbf{L}_1$-space is defined as the set of absolute-value integrable signals, i.e., $\mathbf{L}_1 = \{u(t) \in \mathbf{R} : \int_0^{+\infty} |u(t)|\, dt < \infty\}$. The $\mathbf{L}_1$-norm of a signal $u \in \mathbf{L}_1$, denoted $\|u\|_1$, is given by

$$\|u\|_1 = \int_0^{+\infty} |u(t)|\, dt$$

this norm can be used, for instance, to measure a consumption. In the case of multidimensional signals $u(t) = (u_1(t), \ldots, u_{n_u}(t))^T \in \mathbf{L}_1^{n_u}$ with $u_i(t) \in \mathbf{L}_1$ $i = 1, \ldots, n_u$, the norm is given by

$$\|u\|_1 = \int_0^{+\infty} \sum_{i=1}^{n_u} |u_i(t)|\, dt = \sum_{i=1}^{n_u} \|u_i(t)\|_1$$

$\mathbf{L}_2$-*Space and* $\mathbf{L}_2$-*Norm*

The $\mathbf{L}_2$-space is defined as the set of square integrable signals, i.e., we have $\mathbf{L}_2 = \{u(t) \in \mathbf{R} : \int_0^{+\infty} u(t)^2\, dt < \infty\}$. The $\mathbf{L}_2$-norm of a signal $u \in \mathbf{L}_2$, denoted $\|u\|_2$, is given by

$$\|u\|_2 = \left(\int_0^{+\infty} u(t)^2\, dt\right)^{1/2}$$

the square of this norm represents the total energy contained in the signal. According to *Parseval's theorem*,[2] the $\mathbf{L}_2$-norm of a signal $u \in \mathbf{L}_2$ can be calculated in the frequency-domain as follows:

$$\|u\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|U(j\omega)\right|^2 d\omega\right)^{1/2}$$

where $U(j\omega)$ is the *Fourier transform* of the signal $u(t)$.

In the case of multidimensional signals $u(t) = (u_1(t), \ldots, u_{n_u}(t))^T \in \mathbf{L}_2^{n_u}$ with $u_i(t) \in \mathbf{L}_2$ $i = 1, \ldots, n_u$, the norm is given by

$$\|u\|_2 = \left(\int_0^{+\infty} u(t)^T u(t)\, dt\right)^{\frac{1}{2}} = \left(\int_0^{+\infty} \sum_{i=1}^{n_u} u_i(t)^2\, dt\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n_u} \|u_i\|_2^2\right)^{\frac{1}{2}}$$

$\mathbf{L}_\infty$-*Space and* $\mathbf{L}_\infty$-*Norm*

The $\mathbf{L}_\infty$-space is defined as the set of signals bounded in amplitude, i.e., $\mathbf{L}_\infty = \{u(t) \in \mathbf{R} : \sup_{t\geq 0} |u(t)| < \infty\}$. The $\mathbf{L}_\infty$-norm of a signal $u \in \mathbf{L}_\infty$, denoted $\|u\|_\infty$, is given by

$$\|u\|_\infty = \sup_{t\geq 0}\left|u(t)\right|$$

this norm represents the maximum value that the signal can take. In the case of multidimensional signals $u(t) \in \mathbf{L}_\infty^{n_u}$ $(u(t) = (u_1(t), \ldots, u_{n_u}(t))^T$ with $u_i(t) \in \mathbf{L}_\infty)$, the norm is given by

$$\|u\|_\infty = \max_{1\leq i\leq n_u}\left(\sup_{t\geq 0}\left|u_i(t)\right|\right) = \max_{1\leq i\leq n_u} \|u_i\|_\infty$$

Extended $\mathbf{L}_p$*-Space*

The $\mathbf{L}_p$-space, $p = 1, 2, \infty$, only includes bounded signals. For instance, the $\mathbf{L}_2$-space only includes signals with bounded energy. In order to also include in our study unbounded signals as well, it is necessary to introduce extended versions of the standard $\mathbf{L}_p$-spaces. For this purpose, consider the *projection function* denoted $P_T(.)$ defined as

$$P_T\big(u(t)\big) = u_T(t) = \begin{cases} u(t), & t \le T \\ 0, & t > T \end{cases}$$

where T is a given time interval over which the signal is considered. The *extended* $\mathbf{L}_{pe}$*-space*, $p = 1, 2, \infty$, is then defined as the space of piecewise continuous signals $u : \mathbf{R}_+ \to \mathbf{R}^m$ such that $u_T \in \mathbf{L}_p$.

RMS-Value

Some signals are of special interest for system analysis and synthesis. This is the case, for instance, of the sinusoidal signal $u(t) = A \sin(\omega t + \varphi)$. However, this signal

is not square integrable and is often called an *infinite energy signal*. A very common measurement of the size of an infinite energy signal is the root-mean-square (RMS) value. The *RMS-value* of a given signal $u(t)$ is defined as

$$u_{\mathrm{rms}} = \left(\lim_{T\to\infty} \frac{1}{T} \int_0^T u(t)^2 \, dt \right)^{1/2}$$

The square of this quantity represents the average power of the signal. The RMS-value of a given signal $u(t)$ can be also computed in the frequency domain as follows:

$$u_{\mathrm{rms}} = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_u(\omega) \, d\omega \right)^{1/2}$$

where $S_u(\omega)$ is the *power spectral density*[3] (PSD), which represents the way in which the average power of the signal $u(t)$ is distributed over the frequency range. In the case of multidimensional signals $u(t) = (u_1(t), \ldots, u_{n_u}(t))^T$, the RMS-value of the vector signal $u(t)$ is given by

$$u_{\mathrm{rms}} = \left(\lim_{T\to\infty} \frac{1}{T} \int_0^T u(t)^T u(t) \, dt \right)^{1/2}$$

The RMS-value of a given signal vector $u(t)$ can also be computed in the frequency domain as follows:

$$u_{\text{rms}} = \left(\text{Trace}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_u(\omega)\,d\omega\right)\right)^{1/2}$$

where $S_u(\omega)$ is the power spectral density matrix[4] of the signal vector $u(t)$.

LTI Systems

Broadly speaking, a system can be seen as a device that associates to a given input signal $u(t)$, an output signal $y(t)$. In this book, for tractability reasons, we consider the particular class of linear time invariant finite-dimensional systems or LTI-system

for short. The so-called *state-space representation* of this kind of system is defined as follows:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

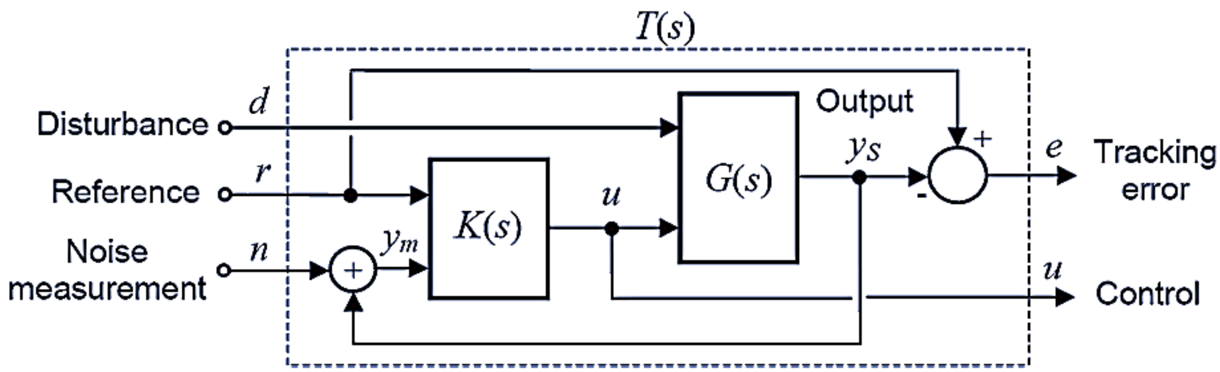
where $u \in \mathbf{R}^{n_u}$ is the input vector, $y \in \mathbf{R}^{n_y}$ is the output vector, $x \in \mathbf{R}^{n_x}$ is the state vector, and A, B, C, D are constant matrices of appropriate dimension.

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$y(t) = Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$

we obtain the input/output relation

$$Y(s) = G(s)U(s), \quad G(s) = C(sI - A)^{-1}B + D$$



System Norms

Given an LTI-system an important issue is to characterize, in some sense, the amplification (or attenuation) introduced by the system for a given input signal. To emphasize the importance of this issue, consider the control problem shown in Fig. 2.2 where $K(s)$ is the controller to be designed and $G(s)$ is the transfer matrix of the system to be controlled.

The objective is to determine the controller $K(s)$ to obtain a low tracking error and a control signal compatible with the possibility of the plant (i.e. the control signal must be admissible by the system) despite the external influences r, d, and n. One way to evaluate the performance of the closed-loop system is to measure the gain provided by the system T, between the inputs (r, d and n) and the outputs e

and u:

$$\begin{bmatrix} e \\ u \end{bmatrix} = T(s) \begin{bmatrix} r \\ d \\ n \end{bmatrix}$$

Good performance is then obtained if the transfer matrix $T(s)$ is small or, more specifically, if the gain of $T(s)$ is small. The word "gain" must be understood here as a measurement of the size of the matrix $T(s)$.

The gain of a system quantifies the amplification provided by the system between the inputs and the outputs. This notion of gain needs to be defined more accurately, this is the subject of the next section on $\mathbf{H}_2$ and $\mathbf{H}_\infty$ norms of a system.

Definition of the $\mathbf{H}_2$*-Norm and* $\mathbf{H}_\infty$*-Norm of a System*

Let $G(s)$ be the transfer function of a stable single input single output (SISO) LTI-system of input $u(t)$ and output $y(t)$. We know that $G(s)$ is the Laplace transform of the impulse response $g(t)$ of the system, we define the $\mathbf{H}_2$-norm of $G(s)$ as the $\mathbf{L}_2$-norm of its impulse response:

$$\|G\|_2 = \left(\int_0^{\infty} g(t)^2 \, dt \right)^{1/2} = \|g\|_2$$

Note that the previous norm is defined for a particular signal which is here the Dirac impulse $\delta(t)$. According to Parseval's theorem the $\mathbf{H}_2$ norm is defined in the frequency domain as follows:

$$\|G\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| G(j\omega) \right|^2 d\omega \right)^{1/2}$$

We can define the gain provided by the system for a given particular input as the ratio of the $\mathbf{L}_2$-norm of the output signal to the $\mathbf{L}_2$-norm of the input signal $\|G\|_{\text{gain}} = \|Gu\|_2 / \|u\|_2$, with $\|u\|_2 \neq 0$. For obvious reason, this gain is often referred to as the $\mathbf{L}_2$-*gain* of the system. Instead of evaluating the $\mathbf{L}_2$-gain for a particular input, one can also determine the greatest possible $\mathbf{L}_2$-gain over the set of square integrable signals, this is the definition of the $\mathbf{H}_\infty$-norm of a system

$$\|G\|_\infty = \sup_{\substack{u \in \mathbf{L}_2 \\ \|u\|_2 \neq 0}} \frac{\|Gu\|_2}{\|u\|_2}$$

This quantity represents the largest possible $\mathbf{L}_2$-gain provided by the system. For a MIMO system with n_u inputs and n_y outputs, the $\mathbf{H}_\infty$-norm is defined as

$$\|G\|_\infty = \sup_{\substack{u \in \mathbf{L}_2^{n_u} \\ \|u\|_2 \neq 0}} \frac{\|Gu\|_2}{\|u\|_2} \quad \text{with } y \in \mathbf{L}_2^{n_y}$$

Singular Values and $\mathbf{H}_2$, $\mathbf{H}_\infty$-Norms

Let $G(s)$ be a stable and strictly proper transfer matrix[13] of dimension $p \times m$. The set of stable and strictly proper transfer matrices is denoted $\mathbf{RH}_2^{n_y \times n_u}$. For any transfer matrix $G(s) \in \mathbf{RH}_2^{n_y \times n_u}$, we define the $\mathbf{H}_2$-norm as[14]

$$\left\| G(s) \right\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{Trace}\big(G(j\omega) G^*(j\omega) \big) \, d\omega \right)^{1/2}$$

this norm can be also expressed using the singular values:[15]

$$\left\| G(s) \right\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{i=1}^{\min(m,p)} \sigma_i^2 \big(G(j\omega) \big) \, d\omega \right)^{1/2}$$

The square of the $\mathbf{H}_2$-norm represents the area under the curve of the sum of squared singular values.

Now, consider a stable and proper transfer matrix $G(s)$. The set of stable and proper transfer matrices is noted $\mathbf{RH}_\infty^{n_y \times n_u}$. For any transfer matrix $G(s) \in \mathbf{RH}_\infty^{n_y \times n_u}$ the $\mathbf{H}_\infty$-norm is defined as

$$\|G(s)\|_\infty = \sup_\omega \bar{\sigma}\left(G(j\omega)\right)$$

This norm represents the largest possible frequency gain, which corresponds to the maximum of the largest singular value of $G(j\omega)$ (see relation (2.35) and Fig. 2.3). In the case of a SISO system, $\|G(s)\|_\infty$ is the maximum of $|G(j\omega)|$

$$\|G\|_\infty = \max_\omega \left|G(j\omega)\right|$$

Energy and power for continuous-time signals

The terms signal energy and signal power are used to characterize a signal. They are not actually measures of energy and power. The definition of signal energy and power refers to any signal $x(t)$, including signals that take on complex values.

Definition

The signal energy in the signal $x(t)$ is

$$E = \int_{-\infty}^{\infty} |\, x(t)|^2 \,\mathrm{d}t \,.$$

The signal power in the signal $x(t)$ is

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\, x(t)|^2 \,\mathrm{d}t \,.$$

If $0 < E < \infty$, then the signal $x(t)$ is called an energy signal. However, there are signals where this condition is not satisfied. For such signals we consider the power. If $0 < P < \infty$, then the signal is called a power signal. Note that the power for an energy signal is zero $(P = 0)$ and that the energy for a power signal is infinite $(E = \infty)$. Some signals are neither energy nor power signals.

Let us consider a periodic signal $x(t)$ with period T_0. The signal energy in one period is

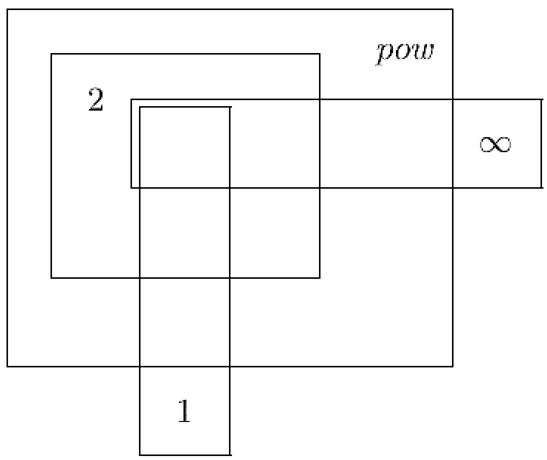
A Venn diagram summarizing the set inclusions is shown in Figure 2.1. Note that the set labeled "*pow*" contains all power signals for which *pow* is finite; the set labeled "1" contains all signals of finite 1-norm; and so on. It is instructive to get examples of functions in all the components of this diagram (Exercise 2).| For example, consider

$$u_1(t) = \begin{cases} 0, & \text{if } t \le 0 \\ 1/\sqrt{t}, & \text{if } 0 < t \le 1 \\ 0, & \text{if } t > 1. \end{cases}$$

This has finite 1-norm:

$$\|u_1\|_1 = \int_0^1 \frac{1}{\sqrt{t}} dt = 2.$$

Its 2-norm is infinite because the integral of $1/t$ is divergent over the interval $[0, 1]$. For the same reason, u_1 is not a power signal. Finally, u_1 is not bounded, so $\|u_1\|_\infty$ is infinite. Therefore, u_1 lives in the bottom component in the diagram.



1. If $\|u\|_2 < \infty$, then u is a power signal with $pow(u) = 0$.
2. If u is a power signal and $\|u\|_\infty < \infty$, then $pow(u) \le \|u\|_\infty$.
3. If $\|u\|_1 < \infty$ and $\|u\|_\infty < \infty$, then $\|u\|_2 \le (\|u\|_\infty \|u\|_1)^{1/2}$, and hence $\|u\|_2 < \infty$.

Assume the system with input "u" and output "y" and stable ans strictly proper transfer function $\hat{G}(s)$

	$u(t) = \delta(t)$	$u(t) = \sin(\omega t)$
$\|y\|_2$	$\|\hat{G}\|_2$	∞
$\|y\|_\infty$	$\|G\|_\infty$	$\|\hat{G}(j\omega)\|$
$pow(y)$	0	$\frac{1}{\sqrt{2}}\|\hat{G}(j\omega)\|$